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On a degenerate mixed-type boundary value problem to the 2-D steady Euler equation

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Abstract

This paper is concerned with the structure of sonic-supersonic solutions near sonic curves arising from the transonic channel flow problems in gas dynamics. Given two smooth curves, one of which is a sonic curve and the other is a characteristic curve, we construct a local classical supersonic solution for the two-dimensional steady Euler equations in the angular region. A novel set of dependent and independent variables are introduced to transform the Euler equations into a linear system with explicitly singularity-regularity structures. We use the iteration method to establish the existence and uniqueness of smooth solutions for the new system and then express the solution in terms of the original variables.

MSC: 35L65; 35L80; 76H05

Keywords: Steady Euler equations; Sonic curve; Degenerate hyperbolic; Classical solution; Characteristic decomposition

1. Introduction

In this paper, we are interested in the existence of classical sonic-supersonic solutions to the steady compressible Euler equations in two dimensions (2-D). The 2-D steady isentropic Euler equations for perfect gases read that

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$$\begin{cases} (\rho u)_x + (\rho v)_y = 0, \\ (\rho u^2 + p)_x + (\rho u v)_y = 0, \\ (\rho u v)_x + (\rho v^2 + p)_y = 0 \end{cases} \tag{1.1}$$

where ρ is the density, (u, v) are the velocity, p is the pressure satisfying the equation of state $p = A\rho^\gamma$ for some positive constant A, and $\gamma > 1$ is the adiabatic index. For the irrotational flows, that is, $u_y = v_x$, system (1.1) can be reduced to

$$\begin{cases} (c^2 - u^2)u_x - uv(u_y + v_x) + (c^2 - v^2)v_y = 0, \\ u_y - v_x = 0, \end{cases} \tag{1.2}$$

which is supplemented with the Bernoulli law

$$q^2 + \frac{2c^2}{\gamma - 1} = \hat{q}^2, \tag{1.3}$$

where $c = \sqrt{p'(\rho)}$ is the sound speed, $q = \sqrt{u^2 + v^2}$ is the flow spend and $\hat{q}$ is a positive integration constant depending on the flow. The eigenvalues of system (1.2) are

$$\Lambda_\pm = \frac{uv \pm c\sqrt{q^2 - c^2}}{u^2 - c^2}. \tag{1.4}$$

It is clear that system (1.2) is of mixed-type: supersonic for $q > c$, subsonic for $q < c$ and sonic for $q = c$. The set of points at which $q = c$ is called the sonic curve.

The motivation of the paper arises from the well-known transonic channel flow problems. In the famous book (Supersonic Flow and Shock Waves, 1948, Page 370, [8]), Courant and Friedrichs described the following transonic phenomena in a duct: *Suppose the duct walls are plane except for a small inward bulge at some section. If the entrance Mach number is not much below the value one, the flow becomes supersonic in a finite region adjacent to the bulge and is again purely subsonic throughout the exit section.* See Fig. 1 for illustration. Similar transonic phenomena occur in many contexts of gas dynamics, such as a flow over an airfoil. The existence of global solutions for the transonic flow problems are open mathematically for a long time and there are many works contributing on this field. In [25], Morawetz showed that there is no smooth solution for the transonic flow problem in general, which means transonic shocks may appear in the flow. The existence of weak solutions were investigated by Morawetz [26] and Chen et al. [3] in the compensated-compactness framework. Chen and Feldman [5] established the existence and stability of the transonic potential flows through an infinite nozzle. In [31], the existence of global solutions in a subsonic-sonic part of the nozzle was verified by Xie and Xin. They also solved in [32] the well-posedness for the subsonic and subsonic-sonic flows with critical mass flux. The transonic shocks arising in supersonic flow past a blunt body or a bounded nozzle has been studied in [7,13,33,34] and references therein. For the related study of the full Euler equations, one may refer, among others, to [2,4,6,10–12,30].

The purpose of the paper is to establish the existence of classical solutions to a degenerate mixed-type boundary value problem for the 2-D steady isentropic irrotational Euler equations (1.2) in the hyperbolic region. Specifically, we consider the problem as follows.

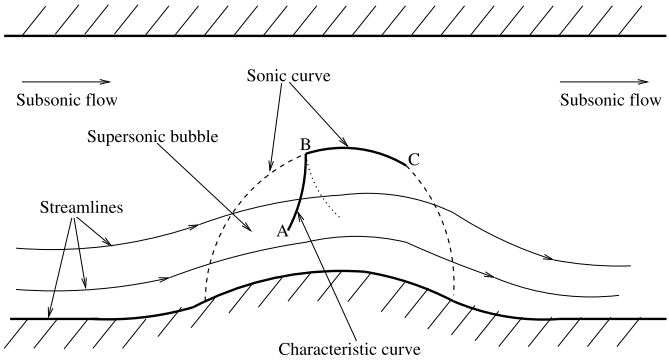

Fig. 1. Transonic phenomena in a duct.

Problem 1. Let $\widehat{BA}$ and $\widehat{BC}$ be two pieces of smooth curves. We assign the boundary data on $\widehat{BA}$ and $\widehat{BC}$ such that $\widehat{BA}$ is a positive characteristic curve and $\widehat{BC}$ is a sonic curve. We look for a classical supersonic solution for (1.2) in the region ABC near point B, see Fig. 1.

Since $\widehat{BC}$ is a sonic curve, that is, $q = c$ on $\widehat{BC}$ and $\widehat{BA}$ is a characteristic curve, then Problem 1 is a degenerate mixed-type boundary value problem, which is also called the degenerate Cauchy-Goursat Problem. To the best of the authors' knowledge, this is the first time to study such kind of degenerate problems for the Euler equations. We hope to extend the local solution near B to global and then construct a transonic shock in the downstream region by assigning appropriate boundary data on $\widehat{BA}$ and $\widehat{BC}$ in future work. In a recent paper [35], Zhang and Zheng established a local classical supersonic solution to (1.2) with degenerate boundary data. The result in this paper can be seen as the development of their result. It is worthwhile to point out that the method introduced in [35] seems to be difficult to deal with the current mixed-type degenerate boundary value problems. In the present paper, we directly establish the convergence of the iterative sequence generated by an integral system. Our approach is inspired by the works done by Berezin [1] and Protter [27] for studying the well-posedness of the Cauchy problem to the second-order linear degenerate hyperbolic equation.

It is well-known that the appropriate change variables may play a crucial role in exploring the properties of solutions near sonic curves for the Euler system. The hodograph method was usually applied to study the transonic flow problems of (1.2), which can linearize the system by switching the roles of (u, v) with (x, y). However, this method is difficult in taking on boundary conditions and in returning back to the original variables due to the sonic degeneracy. The stream and potential functions were introduced as the coordinate system by Kuzmin [19] to discuss the transonic perturbation problems. The variable $\sqrt{q^2 - c^2}$ and the potential function were applied in Zhang and Zheng [35] and other related papers [29,36]. In order to overcome the difficulties caused by the non-existence of potential function, Hu and Li [15] adopted the Mach angle ω and the flow angle θ as the independent variables to construct a classical sonic-supersonic solution for the steady full Euler equations, also see [17] for the study of the self-similar full Euler system. In this paper, we are interested to find that, in terms of $(\cos\omega, \theta)$, the isentropic irrotational Euler equations (1.2) can be transformed as a linear system by introducing a set of dependent variables. We believe that this coordinate transformation will have more applications in the future in researching the transonic flow problems of (1.2).

The rest of the paper is organized as follows. In Section 2, we formulate the problem in terms of the angle variables and then present the main result of the paper. In Section 3, we introduce a set of change variables to linearize the system and then establish the existence of classical solutions to the mixed-type boundary value problem for the linear system. Finally, Section 4 is devoted to returning the solution in the hodograph plane to that in the original physical plane.

2. Formulation of the problem and the main result

In this section, we formulate the problem in terms of the angle variables by deriving their characteristic decompositions. The characteristic decomposition as a powerful tool was first revealed by Dai and Zhang [9] and then developed by Li, Zhang and Zheng [21]. The inclination angle variables of characteristics were chosen as the dependent variables that was invented by Li and Zheng [22] for studying the interaction of rarefaction waves of the 2-D self-similar Euler equations. We refer the reader to [14,16,18,20,23,24,28] and references therein for more relevant applications.

2.1. Characteristic decompositions of the angle variables

For smooth solutions, we can write system (1.2) as

$$\mathbf{A}\mathbf{W}_x + \mathbf{B}\mathbf{W}_y = \mathbf{0}, \tag{2.1}$$

where

$$\mathbf{A} = \begin{pmatrix} c^2 - u^2 & -uv \\ 0 & -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -uv & c^2 - v^2 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

By standard manipulation, the characteristic form of (2.1) is

$$\begin{cases} \partial^+ u + \Lambda_- \partial^+ v = 0, \\ \partial^- u + \Lambda_+ \partial^- v = 0, \end{cases} \qquad \partial^\pm = \partial_x + \Lambda_\pm \partial_y. \tag{2.2}$$

Here $\Lambda_\pm$ is defined in (1.4).

Following Li and Zheng [22], we introduce the flow angle θ and the Mach angle ω as follows

$$\tan\theta = \frac{v}{u}, \quad \sin\omega = \frac{c}{q}. \tag{2.3}$$

Moreover, we denote

$$\alpha := \theta + \omega, \quad \beta := \theta - \omega. \tag{2.4}$$

It is easy to check by the expressions of (1.4) that

$$\tan\alpha = \Lambda_+, \quad \tan\beta = \Lambda_-, \tag{2.5}$$

which means that the angles α and β are the inclination angles of characteristic curves. Furthermore, the velocity (u, v) can be expressed by the angle variables

$$u = c\frac{\cos\theta}{\sin\omega}, \qquad v = c\frac{\sin\theta}{\sin\omega}. \tag{2.6}$$

In addition, we introduce the following weighted directional derivatives

$$\begin{aligned} \bar{\partial}^+ &= \cos\alpha\partial_x + \sin\alpha\partial_y, \quad & \bar{\partial}^- &= \cos\beta\partial_x + \sin\beta\partial_y, \\ \bar{\partial}^0 &= \cos\theta\partial_x + \sin\theta\partial_y, \quad & \bar{\partial}^\perp &= -\sin\theta\partial_x + \cos\theta\partial_y. \end{aligned} \tag{2.7}$$

Thus one has the relations

$$\begin{cases} \partial_x = -\frac{\sin\beta\bar{\partial}^+ - \sin\alpha\bar{\partial}^-}{\sin(2\omega)}, \\ \partial_y = \frac{\cos\beta\bar{\partial}^+ - \cos\alpha\bar{\partial}^-}{\sin(2\omega)}, \end{cases} \qquad \begin{cases} \bar{\partial}^0 = \frac{\bar{\partial}^+ + \bar{\partial}^-}{2\cos\omega}, \\ \bar{\partial}^\perp = \frac{\bar{\partial}^+ - \bar{\partial}^-}{2\sin\omega}. \end{cases} \tag{2.8}$$

By performing direct calculations, we obtain a new system in terms of the variables (θ, ω, c) from (2.2) and (2.7)

$$\begin{cases} \bar{\partial}^+\theta + \frac{\cos^2\omega}{\sin^2\omega}\bar{\partial}^+\omega - \frac{\cos\omega}{\sin\omega}\frac{\bar{\partial}^+c}{c} = 0, \\ \bar{\partial}^-\theta - \frac{\cos^2\omega}{\sin^2\omega}\bar{\partial}^-\omega - \frac{\cos\omega}{\sin\omega}\frac{\bar{\partial}^-c}{c} = 0. \end{cases} \tag{2.9}$$

Moreover, we put (2.6) into the Bernoulli law (1.3) and differentiate the resulting to get

$$\frac{\bar{\partial}^\pm c}{c} = \frac{\kappa}{\sin\omega(\kappa + \sin^2\omega)}\bar{\partial}^\pm \sin\omega. \tag{2.10}$$

Inserting (2.10) into (2.9) yields the system in terms of the variables (θ, ω)

$$\begin{cases} \bar{\partial}^+\theta + \frac{\cos\omega}{\kappa + \sin^2\omega}\bar{\partial}^+\varpi = 0, \\ \bar{\partial}^-\theta - \frac{\cos\omega}{\kappa + \sin^2\omega}\bar{\partial}^-\varpi = 0, \end{cases} \tag{2.11}$$

where $\varpi = \sin\omega$. Here and below, the mixed variables are used in a system for convenience.

In order to get a linear system later, we further introduce a new variable

$$\Xi = \frac{1}{4\kappa}\ln\left(\frac{\varpi^2}{\kappa + \varpi^2}\right). \tag{2.12}$$

Then we have by (2.11)

$$\begin{cases} \bar{\partial}^+\theta + \sin(2\omega)\bar{\partial}^+\Xi = 0, \\ \bar{\partial}^-\theta - \sin(2\omega)\bar{\partial}^-\Xi = 0. \end{cases} \tag{2.13}$$

According to the definition of Ξ gives the relationships

$$\bar{\partial}^i\varpi = 2\varpi(\kappa + \varpi^2)\bar{\partial}^i\Xi, \quad i = 0, \pm. \tag{2.14}$$

Making use of the commutator relation between $\bar{\partial}^+$ and $\bar{\partial}^-$ [22]

$$\bar{\partial}^-\bar{\partial}^+ - \bar{\partial}^+\bar{\partial}^- = \frac{\cos(2\omega)\bar{\partial}^-\alpha - \bar{\partial}^+\beta}{\sin(2\omega)}\bar{\partial}^+ + \frac{\cos(2\omega)\bar{\partial}^+\beta - \tilde{\partial}^-\alpha}{\sin(2\omega)}\bar{\partial}^-, \tag{2.15}$$

we derive by (2.13)

$$\begin{cases} \bar{\partial}^- U = \frac{\kappa U}{\cos^2\omega}(U - \cos(2\omega)V) + \frac{U}{\cos^2\omega}(U + V\cos^2(2\omega)), \\ \bar{\partial}^+ V = \frac{\kappa V}{\cos^2\omega}(V - \cos(2\omega)U) + \frac{V}{\cos^2\omega}(V + U\cos^2(2\omega)), \end{cases} \tag{2.16}$$

where $U = \bar{\partial}^+\Xi$ and $V = \bar{\partial}^-\Xi$.

2.2. *The boundary data and the main result*

We now specify boundary conditions for Problem 1. Let $\widehat{BC} : y = \varphi(x)$ $(x \in [x_B, x_C])$ be a smooth curve. We assume that the curve $\widehat{BC}$ and the boundary values $(\theta, \varpi)|_{\widehat{BC}} = (\hat{\theta}, \hat{\varpi})(x)$ satisfy

$$\varphi(x) \in C^3([x_B, x_C)), \quad \hat{\theta}(x) \in C^3([x_B, x_C)), \quad \hat{\varpi}(x) = 1. \tag{2.17}$$

From (2.17), we see that the curve $\widehat{BC}$ is a smooth sonic curve. Let $\widehat{AB} : x = \psi(y)$ $(y \in [y_A, y_B])$ be a smooth curve satisfying $x_B = \psi(y_B)$. We suppose that the curve $\widehat{AB}$ and the boundary values $(\theta, \varpi)|_{\widehat{AB}} = (\tilde{\theta}, \tilde{\varpi})(y)$ satisfy

$$\begin{gathered} \psi(y) \in C^4((y_A, y_B]), \quad \tilde{\theta}(y) = \operatorname{arccot}\psi'(y) - \arcsin\tilde{\varpi}(y), \quad \tilde{\varpi}(y_B) = 1, \\ \arcsin\tilde{\varpi}(y) - G(\tilde{\varpi}(y)) = \operatorname{arccot}\psi'(y) - [\hat{\theta}(x_B) + G(\hat{\varpi}(x_B))], \end{gathered} \tag{2.18}$$

where $G = G(\varpi)$ is defined as

$$G(\varpi) = \int \frac{\sqrt{1-\varpi^2}}{\kappa + \varpi^2}\, \mathrm{d}\varpi.$$

It follows from (2.18) that the curve $\widehat{BA}$ is a positive characteristic and the point B is sonic. By the definition of $G(\varpi)$, a unique function $\tilde{\varpi}(y) \in C^3((y_A, y_B])$ can be determined by the last equation of (2.18), which is the compatibility condition with the first equation of (2.11).

The main conclusion of the paper is stated in the following theorem.

Theorem 2.1. *Let the boundary conditions* (2.17) *and* (2.18) *hold. Assume that* $\psi''(y_B) < 0$, $\hat{\theta}'(x_B) < 0$ *and* $(\varphi'\sin\hat{\theta} + \cos\hat{\theta})(x_B) > 0$. *Then system* (2.11) *with the boundary data* $(\theta, \varpi)|_{\widehat{BC}} = (\hat{\theta}, \hat{\varpi})(x)$ *and* $(\theta, \varpi)|_{\widehat{BA}} = (\tilde{\theta}, \tilde{\varpi})(y)$ *admits a classical supersonic solution* (θ, ϖ) *in the region* ABC *near the point* B.

2.3. *The boundary information for* (U, V)

In this subsection, we derive the information of (U, V) on the boundary $\widehat{BA}$ and $\widehat{BC}$ from the boundary data of (θ, ϖ).

We first check the compatibility conditions of (2.11) at the point B. It is clear that $\hat{\varpi}(B) = 1 = \tilde{\varpi}(B)$. Moreover, we find by (2.18) that

$$\begin{aligned}\tilde{\theta}(B) &= \operatorname{arccot}\psi'(B) - \arcsin\tilde{\varpi}(B)\\ &= \hat{\theta}(B) + G(\hat{\varpi}(B)) - G(\tilde{\varpi}(B)) = \hat{\theta}(B).\end{aligned}$$

Thus we have $(\hat{\theta}, \hat{\varpi})|_B = (\tilde{\theta}, \tilde{\varpi})|_B$. Now we claim that the second equation of (2.11) holds at the point B. Since $\cos\omega(B) = 0$, then the second equation of (2.11) reduces to $\bar{\partial}^-\theta = 0$ at B. Using (2.18) again gives

$$\tilde{\theta}'(y) = -\frac{\sqrt{1-\tilde{\varpi}(y)}}{\kappa + \tilde{\varpi}^2(y)}\tilde{\varpi}'(y), \tag{2.19}$$

from which one has $\tilde{\theta}'(B) = 0$, which means that $\bar{\partial}^+\theta = 0$ at B. Recalling the definition of $\partial^\pm$ leads to

$$\begin{aligned}\bar{\partial}^-\theta(B) &= \cos(\theta(B) - \omega(B))\theta_x(B) + \sin(\theta(B) - \omega(B))\theta_y(B)\\ &= \sin\theta(B)\theta_x(B) - \cos\theta(B)\theta_y(B)\\ &= -[\cos(\theta(B) + \omega(B))\theta_x(B) + \sin(\theta(B) + \omega(B))\theta_y(B)] = -\bar{\partial}^+\theta(B) = 0.\end{aligned}$$

Next we discuss the information of $(\bar{\partial}^+\Xi, \bar{\partial}^-\Xi)$ on $\widehat{BA}$ and $\widehat{BC}$. Since $\widehat{BA}$ is a positive characteristic, then we have (2.14)

$$\begin{aligned}U|_{\widehat{BA}} = \bar{\partial}^+\Xi|_{\widehat{BA}} &= \left.\frac{\bar{\partial}^+\varpi}{2\varpi(\kappa+\varpi^2)}\right|_{\widehat{BA}} = \frac{\sin\alpha|_{\widehat{BA}}}{2\tilde{\varpi}(\kappa+\tilde{\varpi}^2)}\tilde{\varpi}'\\ &= \frac{\tilde{\varpi}'}{2\tilde{\varpi}(\kappa+\tilde{\varpi}^2)\sqrt{1+(\psi')^2}} =: \tilde{b}_0(y).\end{aligned} \tag{2.20}$$

We see by (2.18) that $\tilde{b}_0 \in C^2((y_A, y_B])$. Furthermore, we recall (2.8) to see that $\bar{\partial}^+\Xi + \bar{\partial}^-\Xi = 2\cos\omega\bar{\partial}^0\Xi$, which indicates that $\bar{\partial}^+\Xi = -\bar{\partial}^-\Xi$ on the sonic curve $\widehat{BC}$. On the other hand, it follows by adding the two equations in (2.13) that

$$\bar{\partial}^+\Xi - \bar{\partial}^-\Xi = -\frac{\bar{\partial}^+\theta + \bar{\partial}^-\theta}{\sin(2\omega)} = -\frac{\bar{\partial}^0\theta}{\sin\omega},$$

from which we obtain

$$U|_{\widehat{BC}} = \bar{\partial}^+\Xi|_{\widehat{BC}} = -\left.\frac{\bar{\partial}^0\theta}{2}\right|_{\widehat{BC}}, \quad V|_{\widehat{BC}} = \bar{\partial}^-\Xi|_{\widehat{BC}} = \left.\frac{\bar{\partial}^0\theta}{2}\right|_{\widehat{BC}}. \tag{2.21}$$

To derive the data $\bar{\partial}^0\theta$ on $\widehat{BC}$, we subtract the two equations in (2.13) to get $\bar{\partial}^\perp\theta = 2(U - V)\cos\omega$, which means that $\bar{\partial}^\perp\theta|_{\widehat{BC}} = 0$. Recalling the fact $\theta(x, \varphi(x)) = \hat{\theta}(x)$ yields

$$\theta_x(x, \varphi(x)) = \frac{\hat{\theta}'\cos\hat{\theta}}{\sin\hat{\theta}\varphi' + \cos\hat{\theta}}, \quad \theta_y(x, \varphi(x)) = \frac{\hat{\theta}'\sin\hat{\theta}}{\sin\hat{\theta}\varphi' + \cos\hat{\theta}},$$

which combined with (2.21) and the definition of $\bar{\partial}^0$ acquire

$$U|_{\widehat{BC}} = -V|_{\widehat{BC}} = -\frac{\bar{\partial}^0\theta}{2}|_{\widehat{BC}} = -\frac{\hat{\theta}'}{2(\cos\hat{\theta} + \varphi'\sin\hat{\theta})} =: -\hat{a}_0(x). \tag{2.22}$$

It follows by (2.17) that $\hat{a}_0 \in C^2([x_B, x_C))$. Moreover, from the above derivation process, we know that $\tilde{b}_0(y_B) = -\hat{a}_0(x_B)$. In addition, it is not difficult to prove by continuity that there exists two small constants $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $-\hat{a}_0(x) \geq \varepsilon_0$ for any $x \in [x_B, x_B + \delta_0]$ and $\tilde{b}_0(y) \geq \varepsilon_0$ for any $y \in [y_B - \delta_0, y_B]$. Since we only consider the existence of solutions near B, then we may assume, without loss of generality,

$$\begin{array}{lll} \hat{\theta}'(x) \leq -\varepsilon_0, & \hat{a}_0(x) \leq -\varepsilon_0, & \forall\, x \in [x_B, x_C), \\ \psi''(y) \leq \varepsilon_0, & \tilde{b}_0(y) \geq \varepsilon_0, & \forall\, y \in (y_A, y_B]. \end{array} \tag{2.23}$$

Otherwise, we can use the points C_1 and A_1 instead of C and A, respectively, such that (2.23) holds on $\widehat{BC_1}$ and $\widehat{BA_1}$.

3. Solutions in a partial hodograph plane

In this section, we introduce a set of change variables to transform the nonlinear system (2.16) to a linear singular equations and then solve this linear problem by showing the convergence of the iterative sequence.

3.1. Reformulated problem in a partial hodograph plane

In this subsection, we reformulate the problem into a new problem by introducing a partial hodograph transformation.

Set

$$t = \cos\omega(x, y), \quad r = \theta(x, y). \tag{3.1}$$

Performing a direct calculation, we achieve the Jacobian of the transformation (3.1)

$$J := \frac{\partial(t, r)}{\partial(x, y)} = \sin\omega(\bar{\partial}^+\omega\bar{\partial}^-\Xi + \bar{\partial}^-\omega\bar{\partial}^+\Xi) = \frac{4(1-t^2)(\kappa+1-t^2)}{t}UV, \tag{3.2}$$

which together with (2.23) obtains that $J \neq 0$ away from the boundary curve $\widehat{BA} \cup \widehat{BC}$.

In terms of the new coordinates (t, r), one has

$$\bar{\partial}^+ = -\frac{2F}{t}U\partial_t - 2\sqrt{1-t^2}Ut\partial_r, \quad \bar{\partial}^- = -\frac{2F}{t}V\partial_t + 2\sqrt{1-t^2}Vt\partial_r, \tag{3.3}$$

where $F = F(t) = (1-t^2)(\kappa+1-t^2) > 0$. Hence, system (2.16) can be transformed to a new closed semi-linear system of (U, V)

$$\begin{cases} U_t - \frac{\sqrt{1-t^2}t^2}{F}U_r = -\frac{(\kappa+1)U}{2FV}\frac{U+V}{t} + \frac{(\kappa+2-2t^2)}{F}Ut, \\ V_t + \frac{\sqrt{1-t^2}t^2}{F}V_r = -\frac{(\kappa+1)V}{2FU}\frac{U+V}{t} + \frac{(\kappa+2-2t^2)}{F}Vt \end{cases} \tag{3.4}$$

We next discuss the boundary data for system (3.4) in the (t, r) coordinates. Due to the assumption $\hat{\theta}' < 0$ on $\widehat{BC}$, the smooth function $r = \hat{\theta}(x)$ is strictly decreasing, which implies that there exists an inverse function, denoted by $x = \hat{x}(r)$ $(r \in (r_1, r_2])$, where $r_1 = \hat{\theta}(x_C)$ and $r_2 = \hat{\theta}(x_B)$. It is clear that the sonic curve $\widehat{BC}$ on the (x, y)-plane is transformed to a segment, called $\widehat{B'C'}$, on $t = 0$ with $r \in (r_1, r_2]$ on the (t, r)-plane. On this segment, we define the function $\hat{a}_0(r) = \hat{a}_0(\hat{x}(r))$ and then the boundary data of (U, V) on $\widehat{B'C'}$ are $(U, V)(0, r) = (-\hat{a}_0, \hat{a}_0)(r)$ for $r \in (r_1, r_2]$. We now consider the curve $\widehat{BA}$. Due to the assumption $\psi''(y) < 0$ on $(y_A, y_B]$, it is easy to obtain that $\tilde{\varpi}'(y) > 0$ and then $\tilde{\theta}' < 0$ on $\widehat{BA} \setminus \{B\}$ by (2.19). Thus, the smooth function $r = \tilde{\theta}(y)$ is strictly decreasing and then an inverse function $y = \tilde{y}(r)$ exists on $[r_2, r_3)$, where $r_3 = \tilde{\theta}(y_A)$. We denote the corresponding curve of $\widehat{BA}$ on the (t, r)-plane by $\widehat{B'A'}$ and the boundary data of U on this curve is $U|_{\widehat{B'A'}} = \tilde{b}_0(\tilde{y}(r))$. Moreover, we have

Lemma 3.1. *The curve $\widehat{B'A'}$ is a positive characteristic curve of system* (3.4) *defined by*

$$r = r_2 + \int_0^t \frac{\sqrt{1-t^2}t^2}{F(t)}\,\mathrm{d}t =: \bar{r}(t). \tag{3.5}$$

Proof. We differentiate the equality $x(t, r) = \psi(y(t, r))$ with respect to t and apply the fact $\psi' = \cot(\theta + \omega)$ to obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\cot(\theta+\omega)y_t - x_t}{x_r - \cot(\theta+\omega)y_r},$$

which combined with the transformation $(x, y) \mapsto (t, r)$ gives

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}t} &= -\frac{\cot(\theta+\omega)\theta_x + \theta_y}{\sin\omega(\omega_y + \cot(\theta+\omega)\omega_x)}\\ &= -\frac{\bar{\partial}^+\theta}{\sin\omega\bar{\partial}^+\omega} = \frac{\cos^2\omega}{\sin\omega(\kappa + \sin^2\omega)} = \frac{\sqrt{1-t^2}}{F(t)}t^2,\end{aligned}$$

which means that $\widehat{B'A'}$ is a positive characteristic curve of system (3.4). The expression (3.5) is straightforward and the proof of the lemma is complete. □

In addition, we see from (3.4) that, for smooth solutions, there holds

$$U_t|_{t=0} = \left.\frac{U+V}{2t}\right|_{t=0}, \quad V_t|_{t=0} = \left.\frac{U+V}{2t}\right|_{t=0}. \tag{3.6}$$

By the definitions of (U, V), we note that the term $(U + V)/(2t)$ in the (t, r) coordinates corresponds to the term $\bar{\partial}^0\Xi$ in the (x, y) coordinates. So we first derive the boundary value $\bar{\partial}^0\Xi|_{\widehat{BC}}$. It follows from (2.14) that

$$\bar{\partial}^0\Xi|_{\widehat{BC}} = \frac{1}{2(\kappa+1)}(\bar{\partial}^0\varpi)|_{\widehat{BC}}. \tag{3.7}$$

According to (2.11) and (2.8), we get

$$\bar{\partial}^0\theta + \frac{\varpi}{\kappa+\varpi^2}\bar{\partial}^\perp\varpi = 0,$$

from which and (2.22), one has

$$\bar{\partial}^\perp\varpi|_{\widehat{BC}} = -(\kappa+1)\bar{\partial}^0\theta|_{\widehat{BC}} = -2(\kappa+1)\hat{a}_0(x),$$

which together with the fact $\varpi \equiv 1$ on $\widehat{BC}$ gives

$$(\partial_x\varpi)|_{\widehat{BC}} = \frac{2(\kappa+1)\hat{a}_0\varphi'}{\cos\hat{\theta}+\varphi'\sin\hat{\theta}}(x), \quad (\partial_y\varpi)|_{\widehat{BC}} = \frac{-2(\kappa+1)\hat{a}_0}{\cos\hat{\theta}+\varphi'\sin\hat{\theta}}(x).$$

Putting the above into (3.7) yields

$$\bar{\partial}^0\Xi|_{\widehat{BC}} = \frac{\varphi'\cos\hat{\theta}-\sin\hat{\theta}}{\varphi'\sin\hat{\theta}+\cos\hat{\theta}}\hat{a}_0(x) =: \hat{a}_1(x). \tag{3.8}$$

Denote $\hat{a}_1(r) = \hat{a}_1(\hat{x}(r))$, then by (3.6) we have the boundary data of (U_t, V_t) on $\widehat{B'C'}$ as $U_t|_{\widehat{B'C'}} = V_t|_{\widehat{B'C'}} = \hat{a}_1(r)$. It is easy to see that $\hat{a}_1$ is a C^2 function on $(r_1, r_2]$.

In sum, we study the system (3.4) with the following boundary conditions

$$\begin{aligned} U(0,r) = -\hat{a}_0(r),\ V(0,r) = \hat{a}_0(r),\ U_t(0,r) = V_t(0,r) = \hat{a}_1(r), &\quad \text{on } \widehat{B'C'},\\ U(t,\bar{r}(t)) = \bar{b}_0(t), &\quad \text{on } \widehat{B'A'}, \end{aligned} \tag{3.9}$$

where $\bar{b}_0(t) = \tilde{b}_0(\tilde{y}(\bar{r}(t)))$. Thanks to (2.17), (2.18) and (2.23), we know that the functions $\hat{a}_0$, $\hat{a}_1$ and $\bar{b}_0$ satisfy

$$\begin{aligned} \hat{a}_0 \in C^2((r_1,r_2]),\ \hat{a}_1 \in C^2((r_1,r_2]),\ \bar{b}_0 \in C^2([0,t_0)),\\ \hat{a}_0 \le -\varepsilon_0, \quad \bar{b}_0 \ge \varepsilon_0, \end{aligned} \tag{3.10}$$

where $t_0 = \sqrt{1-\tilde{\varpi}^2(A)}$. Furthermore, it is easily checked by system (3.4) and the definitions of $\hat{a}_0$, $\hat{a}_1$ and $\bar{b}_0$ that the compatibility conditions at the point B' hold, i.e., $\bar{b}_0(0) = -\hat{a}_0(r_2)$ and $\bar{b}_0'(0) = \hat{a}_1(r_2)$. Hence the problem in terms of (t,r)-plane can be reformulated as follows.

Problem 2. Under the assumption (3.10), we seek a local classical solution for system (3.4) with boundary conditions (3.9) in the region $t > 0$ near the point $B'(0, r_2)$.

For Problem 2, we have the following existence theorem, to be proved in the next subsection.

Theorem 3.1. *Let* (3.10) *be satisfied. Then system* (3.4) *with boundary conditions* (3.9) *admits a unique classical solution near the point* $B'(0, r_2)$.

3.2. Solutions in the partial hodograph plane

This subsection is devoted to solving Problem 2 in the partial hodograph plane (t, r).

3.2.1. The linear problem

Introduce

$$\overline{U} = \frac{1}{U}, \quad \overline{V} = -\frac{1}{V}. \tag{3.11}$$

We find that system (3.4) can be transformed as a linear system

$$\begin{cases} \overline{U}_t - \frac{\sqrt{1-t^2}t^2}{F}\overline{U}_r = \frac{\overline{U}-\overline{V}}{2t} + \frac{(\kappa+2-t^2)}{2F}(\overline{U}-\overline{V})t - \frac{(\kappa+2-2t^2)}{F}\overline{U}t, \\ \overline{V}_t + \frac{\sqrt{1-t^2}t^2}{F}\overline{V}_r = \frac{\overline{V}-\overline{U}}{2t} - \frac{(\kappa+2-t^2)}{2F}(\overline{U}-\overline{V})t - \frac{(\kappa+2-2t^2)}{F}\overline{V}t. \end{cases} \tag{3.12}$$

Moreover, we preform the following coordinate transformation

$$\tau = t, \quad z = \bar{r}(t) - r, \tag{3.13}$$

to rewrite (3.12) as

$$\begin{cases} \widetilde{U}_\tau + \frac{2\sqrt{1-\tau^2}\tau^2}{F}\widetilde{U}_z = \frac{\widetilde{U}-\widetilde{V}}{2\tau} + \frac{(\kappa+2-\tau^2)}{2F}(\widetilde{U}-\widetilde{V})\tau - \frac{(\kappa+2-2\tau^2)}{F}\widetilde{U}\tau, \\ \widetilde{V}_\tau = \frac{\widetilde{V}-\widetilde{U}}{2\tau} - \frac{(\kappa+2-\tau^2)}{2F}(\widetilde{U}-\widetilde{V})\tau - \frac{(\kappa+2-2\tau^2)}{F}\widetilde{V}\tau. \end{cases} \tag{3.14}$$

Here $F = (1-\tau^2)(\kappa+1-\tau^2)$, $\widetilde{U}(\tau, z) = \overline{U}(\tau, \bar{r}(\tau) - z)$ and $\widetilde{V}(\tau, z) = \overline{V}(\tau, \bar{r}(\tau) - z)$. Corresponding to (3.9), the boundary data of system (3.14) are

$$\begin{aligned} \widetilde{U}(0,z) = \widetilde{V}(0,z) = a_0(z), & \\ \widetilde{U}_\tau(0,z) = -a_1(z),\ \widetilde{V}_\tau(0,z) = a_1(z), & \quad \text{on } \tau = 0,\ 0 \le z < r_2 - r_1, \\ \widetilde{U}(\tau,0) = b_0(\tau), & \quad \text{on } z = 0,\ 0 \le \tau < t_0, \end{aligned} \tag{3.15}$$

where

$$a_0(z) = -\frac{1}{\hat{a}_0(r_2 - z)}, \quad a_1(z) = \frac{\hat{a}_1(r_2 - z)}{\hat{a}_0^2(r_2 - z)}, \quad b_0(\tau) = \frac{1}{\bar{b}_0(\tau)}, \tag{3.16}$$

which are all C^2 smooth functions by (3.10) and satisfy $a_0(0) = b_0(0)$, $b_0'(0) = -a_1(0)$.

We further introduce variables (R, S) to homogenize the boundary conditions

$$R = \widetilde{U} - a_0(z) + a_1(z)\tau, \quad S = \widetilde{V} - a_0(z) - a_1(z)\tau. \tag{3.17}$$

Therefore the boundary values of (R, S) take as

$$\begin{aligned} R(0,z) = S(0,z) = R_\tau(0,z) = S_\tau(0,z) = 0,\ \forall\, z \in [0, r_2 - r_1), \\ R(\tau,0) = b_1(\tau),\ \ \forall\, \tau \in [0, t_0), \end{aligned} \tag{3.18}$$

where the function $b_1(\tau)$ is defined by $b_1(\tau) = b_0(\tau) - a_0(0) + a_1(0)\tau$. It is easily seen by (3.16) that $b_1(0) = b_1'(0) = 0$. In terms of (R, S), the system reads that

$$\begin{cases} R_\tau + \frac{2\sqrt{1-\tau^2}\tau^2}{F} R_z = \frac{R-S}{2\tau} + \frac{\kappa+2-\tau^2}{2F}(R-S)\tau - \frac{\kappa+2-2\tau^2}{F} R\tau + F_1(\tau,z)\tau, \\ S_\tau = \frac{S-R}{2\tau} - \frac{\kappa+2-\tau^2}{2F}(R-S)\tau - \frac{\kappa+2-2\tau^2}{F} S\tau + F_2(\tau,z)\tau, \end{cases} \tag{3.19}$$

where

$$F_1(\tau,z) = -\frac{2\sqrt{1-\tau^2}}{F}(a_0' - a_1'\tau)\tau - \frac{\kappa+2-2\tau^2}{F} a_0 - \frac{a_1\tau^3}{F},$$
$$F_2(\tau,z) = -\frac{\kappa+2-2\tau^2}{F} a_0 + \frac{a_1\tau^3}{F}.$$

It is clear that F_1 and F_2 possess one continuous derivative with respect to z. Hence Problem 2 is transformed to the degenerate linear initial-characteristic problem (3.19) (3.18). The two eigenvalues of (3.19) are

$$\lambda_- = 0, \quad \lambda_+ = \frac{2\sqrt{1-\tau^2}}{F(\tau)}\tau^2. \tag{3.20}$$

We denote by $z = \bar{z}(\tau)$ the positive characteristic passing through the origin, that is,

$$\bar{z}(\tau) = \int_0^\tau \frac{2\sqrt{1-s^2}}{F(s)} s^2 \, \mathrm{d}s.$$

Let Ω be the square domain $\{(\tau,z)|\ 0 \le \tau \le \delta,\ 0 \le z \le \delta\}$ for some constant $\delta > 0$. For any point $(\xi,\eta) \in \Omega$, we integrate the system (3.19) along the characteristics and use (3.18) to obtain a system of integral equations

$$S(\xi,\eta) = \int_0^\xi \left\{ \frac{S-R}{2\tau} - \frac{\kappa+2-\tau^2}{2F}(R-S)\tau - \frac{\kappa+2-2\tau^2}{F} S\tau + F_2\tau \right\}(\tau, z_-(\tau))\, \mathrm{d}\tau,$$
$$R(\xi,\eta) = \begin{cases} \displaystyle\int_0^\xi \left\{ \frac{R-S}{2\tau} + \frac{\kappa+2-\tau^2}{2F}(R-S)\tau - \frac{\kappa+2-2\tau^2}{F} R\tau + F_1\tau \right\}(\tau, z_+(\tau))\, \mathrm{d}\tau, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \eta \ge \bar{z}(\xi), \quad \xi \ge 0, \\ \displaystyle\int_{\xi_1}^\xi \left\{ \frac{R-S}{2\tau} + \frac{\kappa+2-\tau^2}{2F}(R-S)\tau - \frac{\kappa+2-2\tau^2}{F} R\tau + F_1\tau \right\}(\tau, z_+(\tau))\, \mathrm{d}\tau \\ \quad + b_1(\xi_1), \qquad\qquad\qquad\qquad\qquad\qquad 0 \le \eta < \bar{z}(\xi), \quad \xi \ge 0, \end{cases} \tag{3.21}$$

where $z_-(\tau) = z_-(\tau;\xi,\eta)$, $z_+(\tau) = z_+(\tau;\xi,\eta)$ are, respectively, the negative and positive characteristics passing through the point (ξ,η), and $\xi_1 \in [0,\xi)$ is a function of (ξ,η) determined by the following equality

$$\eta = \int_{\xi_1}^{\xi} \frac{2\sqrt{1-\tau^2}}{F(\tau)} \tau^2 \, \mathrm{d}\tau. \tag{3.22}$$

In addition, we note by the fact $\lambda_- = 0$ that $z_-(\tau; \xi, \eta) = \eta$.

We proceed by iterations to establish the existence of solutions of system (3.21). Denote $R^{(0)}(\tau, z) = S^{(0)}(\tau, z) \equiv 0$. We define the quantities $R^{(k)}$ and $S^{(k)}$ $(k \geq 1)$ by the relations

$$\begin{aligned} S^{(k)}(\xi, \eta) = \int_0^{\xi} \Big\{ & \frac{S^{(k-1)} - R^{(k-1)}}{2\tau} - \frac{\kappa + 2 - \tau^2}{2F} (R^{(k-1)} - S^{(k-1)})\tau \\ & - \tfrac{\kappa+2-2\tau^2}{F} S^{(k-1)}\tau + F_2\tau \Big\}(\tau, z_-(\tau)) \, \mathrm{d}\tau, \end{aligned}$$

$$R^{(k)}(\xi, \eta) = \begin{cases} \displaystyle\int_0^{\xi} \Big\{ \frac{R^{(k-1)} - S^{(k-1)}}{2\tau} + \frac{\kappa + 2 - \tau^2}{2F} (R^{(k-1)} - S^{(k-1)})\tau \\ \qquad \displaystyle - \frac{\kappa + 2 - 2\tau^2}{F} R^{(k-1)}\tau + F_1\tau \Big\}(\tau, z_+(\tau)) \, \mathrm{d}\tau, \quad \eta \geq \bar{z}(\xi), \quad \xi \geq 0, \\ \displaystyle\int_{\xi_1}^{\xi} \Big\{ \frac{R^{(k-1)} - S^{(k-1)}}{2\tau} + \frac{\kappa + 2 - \tau^2}{2F} (R^{(k-1)} - S^{(k-1)})\tau \\ \displaystyle - \frac{\kappa + 2 - 2\tau^2}{F} R^{(k-1)}\tau + F_1\tau \Big\}(\tau, z_+(\tau)) \, \mathrm{d}\tau + b_1(\xi_1), \ 0 \leq \eta < \bar{z}(\xi), \ \xi \geq 0. \end{cases} \tag{3.23}$$

We shall show that the sequences $\{(R^{(k)}, S^{(k)})\}$ converge uniformly in the domain Ω for some small $\delta > 0$.

3.2.2. Two key lemmas

We first choose $\delta < \frac{1}{4}$ small enough such that

$$0 < \frac{\kappa + 2 - \tau^2}{(1-\tau^2)(\kappa + 1 - \tau^2)} \leq 2, \quad 0 < \frac{\sqrt{1-\tau^2}}{(1-\tau^2)(\kappa + 1 - \tau^2)} \leq 1, \tag{3.24}$$

hold for $\tau \in [0, \delta]$. We comment that the above upper bounds are just for the convenience of calculation. Throughout the paper, the notation M will denote a constant depending only on the C^2 norms of a_0, a_1 and b_0, which may change from one expression to another. By the expressions of F_1 and F_2, one gets

$$|F_1| + |F_{1z}| + |F_2| + |F_{2z}| \leq M. \tag{3.25}$$

Let $P(\xi, \eta)$ be any point in $\Omega = \{(\tau, z) | \ 0 < \tau \leq \delta, \ 0 < z \leq \delta\}$. Then by the expression of $b_1(\cdot)$, we have if $\eta < \bar{z}(\xi)$,

$$|b_1(\xi_1)| \le \frac{M}{2}\xi_1^2 \le \frac{M}{2}\xi^2. \tag{3.26}$$

Moreover, we can derive the estimate of the distance between $z_+(\tilde{\xi})$ and $z_-(\tilde{\xi})$

$$\begin{aligned} |z_+(\tilde{\xi}) - z_-(\tilde{\xi})| =& |z_+(\tilde{\xi};\xi,\eta) - z_-(\tilde{\xi};\xi,\eta)| \\ \le& |z_+(0;\xi,\eta) - \eta| = \left| \int_0^{\xi} \frac{2\sqrt{1-\tau^2}\tau^2}{F(\tau)}\,\mathrm{d}\tau \right| \le \frac{2}{3}\xi^3 \end{aligned} \tag{3.27}$$

for $\tilde{\xi} \in [0,\xi]$ if $\eta \ge \bar{z}(\xi)$, and

$$\begin{aligned} |z_+(\tilde{\xi}) - z_-(\tilde{\xi})| =& |z_+(\tilde{\xi};\xi,\eta) - z_-(\tilde{\xi};\xi,\eta)| \\ \le& |z_+(\xi_1;\xi,\eta) - \eta| = \left| \int_{\xi_1}^{\xi} \frac{2\sqrt{1-\tau^2}\tau^2}{F(\tau)}\,\mathrm{d}\tau \right| \le \frac{2}{3}\xi^3 \end{aligned} \tag{3.28}$$

for $\tilde{\xi} \in [\xi_1,\xi]$ if $\eta < \bar{z}(\xi)$.

There has

Lemma 3.2. *For all $k \ge 1$ the following inequalities*

$$\begin{cases} \left|R^{(k)}(\xi,\eta)\right|; \left|S^{(k)}(\xi,\eta)\right| \le M\xi^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j, \\ \left|R^{(k)}(\xi,\eta) - S^{(k)}(\xi,\eta)\right| \le M\xi^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j, \end{cases} \tag{3.29}$$

hold in Ω.

Proof. We use the argument of induction to prove the lemma. That is, we first show that all inequalities in (3.29) hold for $n = 1$, and then assuming they all hold for $n = k$ we establish each inequality for $n = k + 1$. The proof is divided into two cases I: $\eta < \bar{z}(\xi)$ and II: $\eta \ge \bar{z}(\xi)$. We just only consider Case I, and Case II can be analyzed analogously.

Thanks to (3.25), we obtain by (3.23)

$$|S^{(1)}(\xi,\eta)| \le \int_0^{\xi} |F_2\tau|\,\mathrm{d}\tau \le \int_0^{\xi} M\tau\,\mathrm{d}\tau \le \frac{M}{2}\xi^2 \le M\xi^2 \sum_{j=0}^{1} \left(\frac{2}{3}\right)^j. \tag{3.30}$$

For the quantity $R^{(1)}$, one has by $\eta < \bar{z}(\xi)$,

$$
\begin{aligned}
|R^{(1)}(\xi,\eta)| &\le \int_{\xi_1}^{\xi} |F_1\tau|\,\mathrm{d}\tau + |b_1(\xi_1)| \le \int_{\xi_1}^{\xi} M\tau\,\mathrm{d}\tau + M\xi^2 \\
&\le \frac{M}{2}(\xi^2-\xi_1^2) + \frac{M}{2}\xi^2 \le M\xi^2 \sum_{j=0}^{1}\left(\frac{2}{3}\right)^j. \qquad (3.31)
\end{aligned}
$$

For estimate $|R^{(k)}(\xi,\eta) - S^{(k)}(\xi,\eta)|$, we first calculate by the expressions of F_1 and F_2

$$
\begin{aligned}
&|F_1(\tau, z_+(\tau;\xi,\eta)) - F_2(\tau, z_-(\tau;\xi,\eta))| \\
\le& \frac{2\sqrt{1-\tau^2}}{F}(|a_0'| + |a_1'|\tau)\tau + \frac{\kappa+2-2\tau^2}{F}|a_0(z_+(\tau;\xi,\eta)) - a_0(z_-(\tau;\xi,\eta))| \\
&+ \frac{|a_1(z_+(\tau;\xi,\eta))| + |a_1(z_-(\tau;\xi,\eta))|}{F}\tau^3 \\
\le& 2(M+M\tau)\tau + 2M|z_+(\tau;\xi,\eta) - z_-(\tau;\xi,\eta)| + 4M\tau^3 \\
\le& 2(M+M\xi)\xi + \frac{4}{3}M\xi^3 + 4M\xi^3 \le 3M\xi \qquad (3.32)
\end{aligned}
$$

for $\tau \in [\xi_1, \xi]$. We now check the estimate of $|R^{(1)}(\xi,\eta) - S^{(1)}(\xi,\eta)|$. Applying (3.32), (3.25) and (3.26), one obtains by a direct calculation

$$
\begin{aligned}
\left|R^{(1)}(\xi,\eta) - S^{(1)}(\xi,\eta)\right| \le& \int_{\xi_1}^{\xi} \left|F_1(\tau, z_+(\tau;\xi,\eta)) - F_2(\tau, z_-(\tau;\xi,\eta))\right| \cdot \tau\,\mathrm{d}\tau \\
&+ \int_0^{\xi_1} |F_2(\tau, z_-(\tau;\xi,\eta))| \cdot \tau\,\mathrm{d}\tau + |b_1(\xi_1)| \\
\le& \int_{\xi_1}^{\xi} 3M\xi\tau\,\mathrm{d}\tau + \int_0^{\xi_1} M\tau\,\mathrm{d}\tau + \frac{1}{2}M\xi^2 \\
\le& \frac{3}{2}M\xi^3 + \frac{1}{2}M\xi^2 + \frac{1}{2}M\xi^2 \le M\xi^2(1+\frac{3}{2}\delta) \le M\xi^2 \sum_{j=0}^{1}\left(\frac{2}{3}\right)^j. \qquad (3.33)
\end{aligned}
$$

We combine (3.30)-(3.31) and (3.33) to see that (3.29) holds for $n=1$. Next we assume that they are valid for $n=k$. Then for $n=k+1$ one finds,

$$
\begin{aligned}
\left|S^{(k+1)}(\xi,\eta)\right| \le \int_0^{\xi} \Bigg| &- \frac{R^{(k)}(\tau, z_-(\tau)) - S^{(k)}(\tau, z_-(\tau))}{2\tau} - \frac{\kappa+2-2\tau^2}{F} S^{(k)}(\tau, z_-(\tau))\tau \\
&- \frac{\kappa+2-\tau^2}{2F}[R^{(k)}(\tau, z_-(\tau)) - S^{(k)}(\tau, z_-(\tau))]\tau + F_2(\tau, z_-(\tau))\tau \Bigg| \,\mathrm{d}\tau
\end{aligned}
$$

$$\leq \int_0^{\xi} \left\{ \frac{|R^{(k)}(\tau, z_-(\tau)) - S^{(k)}(\tau, z_-(\tau))|}{2\tau} + 2|S^{(k)}(\tau, z_-(\tau))|\tau \right.$$
$$\left. + |R^{(k)}(\tau, z_-(\tau)) - S^{(k)}(\tau, z_-(\tau))|\tau + |F_2(\tau, z_-(\tau))|\tau \right\} \mathrm{d}\tau,$$

from which and the induction assumptions, we acquire

$$\begin{aligned}
\left|S^{(k+1)}(\xi,\eta)\right| &\leq \int_0^{\xi} \left\{ \frac{M}{2} \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \tau + 2M \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \tau^3 + M \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \tau^3 + M\tau \right\} \mathrm{d}\tau \\
&= M\xi^2 (\frac{1}{4} + \frac{3}{4}\xi^2) \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j + \frac{1}{2} M\xi^2 \\
&= M\xi^2 \left\{ \frac{1}{2} + (\frac{1}{4} + \frac{3}{4}\xi^2) \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \right\} \leq M \sum_{j=0}^{k+1} \left(\frac{2}{3}\right)^j \xi^2. \qquad (3.34)
\end{aligned}$$

A similar argument for $R^{(k)}$ leads to

$$\begin{aligned}
\left|R^{(k+1)}(\xi,\eta)\right| &\leq \int_{\xi_1}^{\xi} \left\{ \frac{|R^{(k)}(\tau, z_+(\tau)) - S^{(k)}(\tau, z_+(\tau))|}{2\tau} + 2|R^{(k)}(\tau, z_+(\tau))|\tau \right. \\
&\qquad \left. + |R^{(k)}(\tau, z_+(\tau)) - S^{(k)}(\tau, z_+(\tau))|\tau + M\tau \right\} \mathrm{d}\tau + \frac{M}{2}\xi^2 \\
&\leq \int_{\xi_1}^{\xi} \left\{ \frac{M}{2} \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \tau + 3M \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \tau^3 + M\tau \right\} \mathrm{d}\tau + \frac{M}{2}\xi^2 \\
&\leq M\xi^2 \left\{ 1 + (\frac{1}{4} + \frac{3}{4}\xi^2) \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \right\} \leq M \sum_{j=0}^{k+1} \left(\frac{2}{3}\right)^j \xi^2. \qquad (3.35)
\end{aligned}$$

Finally, we consider the estimate of $|R^{(k+1)}(\xi,\eta) - S^{(k+1)}(\xi,\eta)|$. This term is divided into the following three terms

$$\left|R^{(k+1)}(\xi,\eta) - S^{(k+1)}(\xi,\eta)\right| \leq I_1 + I_2 + |b_1(\xi_1)|, \qquad (3.36)$$

where

$$\begin{aligned}
I_1 = \int_{\xi_1}^{\xi} &\left\{ \frac{|(R^{(k)} - S^{(k)})(\tau, z_+(\tau;\xi,\eta))| + |(R^{(k)} - S^{(k)})(\tau, z_-(\tau;\xi,\eta))|}{2\tau} \right. \\
&+ \frac{\kappa + 2 - \tau^2}{2F} \Big[|(R^{(k)} - S^{(k)})(\tau, z_+(\tau;\xi,\eta))| + |(R^{(k)} - S^{(k)})(\tau, z_-(\tau;\xi,\eta))| \Big] \tau
\end{aligned}$$

$$+\frac{\kappa+2-2\tau^2}{F}\Big[|R^{(k)}(\tau,z_+(\tau;\xi,\eta))|+|S^{(k)}(\tau,z_-(\tau;\xi,\eta))|\Big]\tau$$
$$+\Big|F_1(\tau,z_+(\tau;\xi,\eta))-F_2(\tau,z_-(\tau;\xi,\eta))\Big|\tau\Big\}\,\mathrm{d}\tau,$$

and

$$I_2=\int_0^{\xi_1}\Big\{\frac{|(R^{(k)}-S^{(k)})(\tau,z_-(\tau;\xi,\eta))|}{2\tau}+\frac{\kappa+2-\tau^2}{2F}|(R^{(k)}-S^{(k)})(\tau,z_-(\tau;\xi,\eta))|\tau$$
$$+\frac{\kappa+2-2\tau^2}{F}|S^{(k)}(\tau,z_-(\tau;\xi,\eta))|\tau+|F_2(\tau,z_-(\tau;\xi,\eta))|\tau\Big\}\,\mathrm{d}\tau.$$

According to (3.32), it suggests

$$\begin{aligned}I_1&\le\int_{\xi_1}^{\xi}\Big\{M\tau\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+2M\tau^3\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+4M\tau^3\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+3M\xi\cdot\tau\Big\}\,\mathrm{d}\tau\\&\le\frac{M}{2}(\xi^2-\xi_1^2)\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+\frac{3M}{2}(\xi^4-\xi_1^4)\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+\frac{3M}{2}\xi^3.\end{aligned}\tag{3.37}$$

We also have

$$\begin{aligned}I_2&\le\int_0^{\xi_1}\Big\{\frac{M}{2}\tau\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+M\tau^3\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+2M\tau^3\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+M\tau\Big\}\,\mathrm{d}\tau\\&\le\frac{M}{4}\xi_1^2\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+\frac{3M}{4}\xi_1^4\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+\frac{M}{2}\xi_1^2.\end{aligned}\tag{3.38}$$

Putting (3.37) and (3.38) into (3.36) gives

$$\begin{aligned}&\Big|R^{(k+1)}(\xi,\eta)-S^{(k+1)}(\xi,\eta)\Big|\\&\le\Big\{\frac{M}{2}(\xi^2-\xi_1^2)\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+\frac{3M}{2}(\xi^4-\xi_1^4)\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+\frac{3M}{2}\xi^3\Big\}\\&\quad+\Big\{\frac{M}{4}\xi_1^2\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+\frac{3M}{4}\xi_1^4\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j+\frac{M}{2}\xi_1^2\Big\}+\frac{M}{2}\xi^2\\&\le M\xi^2\Big\{\frac{1}{2}+\frac{3}{2}\delta+\Big[\frac{1}{2}-\frac{1}{4}\sum_{j=0}^{1}\Big(\frac{2}{3}\Big)^j\Big]+(\frac{1}{2}+\frac{3}{2}\delta^2)\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j\Big\}\end{aligned}$$

$$\leq M\xi^2\left\{1+(\frac{1}{2}+\frac{3}{32})\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\}\leq M\xi^2\sum_{j=0}^{k+1}\left(\frac{2}{3}\right)^j. \tag{3.39}$$

Summing up (3.34)-(3.35) and (3.39), the induction is complete, which finishes the proof of the lemma. □

Furthermore, we have the following lemma

Lemma 3.3. *For all $k\geq 1$ the inequalities*

$$\begin{cases} \left|R^{(k+1)}(\xi,\eta)-R^{(k)}(\xi,\eta)\right|;\ \left|S^{(k+1)}(\xi,\eta)-S^{(k)}(\xi,\eta)\right|\leq M\xi^2\left(\frac{2}{3}\right)^k, \\ \left|R^{(k+1)}(\xi,\eta)-S^{(k+1)}(\xi,\eta)-R^{(k)}(\xi,\eta)+S^{(k)}(\xi,\eta)\right|\leq M\xi^2\left(\frac{2}{3}\right)^k, \end{cases} \tag{3.40}$$

hold in Ω.

Proof. We use the induction to prove the lemma. It is obvious by (3.29) that each of the inequalities in (3.40) holds for $n=1$. Assume they are valid for $n=k$. We shall check that they are preserved for $n=k+1$. The case $\eta\geq\bar{z}(\xi)$ is considered here and the other case can be discussed in a similar way.

For $n=k+1$, we first obtain by (3.23)

$$\begin{aligned} &\left|S^{(k+1)}(\xi,\eta)-S^{(k)}(\xi,\eta)\right| \\ \leq&\int_0^\xi\left\{\frac{|S^{(k)}-R^{(k)}-S^{(k-1)}+R^{(k-1)}|}{2\tau}+\frac{\kappa+2-\tau^2}{2F}\cdot|R^{(k)}-S^{(k)}-R^{(k-1)}+S^{(k-1)}|\cdot\tau\right. \\ &\left.+\frac{\kappa+2-2\tau^2}{F}\cdot|S^{(k)}-S^{(k-1)}|\cdot\tau\right\}(\tau,z_-(\tau))\,\mathrm{d}\tau, \end{aligned}$$

which together with the induction assumptions and the fact $\delta<\frac{1}{4}$, we see that

$$\begin{aligned} &\left|S^{(k+1)}(\xi,\eta)-S^{(k)}(\xi,\eta)\right| \\ \leq&\int_0^\xi\left\{\frac{M}{2}\tau\left(\frac{2}{3}\right)^{k-1}+M\tau^3\left(\frac{2}{3}\right)^{k-1}+2M\tau^3\left(\frac{2}{3}\right)^{k-1}\right\}\mathrm{d}\tau \\ =&M\xi^2(\frac{1}{4}+\frac{3}{4}\xi^2)\left(\frac{2}{3}\right)^{k-1}\leq M\xi^2(\frac{1}{4}+\frac{3}{4}\delta^2)\left(\frac{2}{3}\right)^{k-1}\leq M\xi^2\left(\frac{2}{3}\right)^k. \end{aligned} \tag{3.41}$$

We similarly have

$$
\begin{aligned}
&\left|R^{(k+1)}(\xi,\eta)-R^{(k)}(\xi,\eta)\right| \\
&\le \int_0^{\xi}\left\{\frac{|R^{(k)}-S^{(k)}-R^{(k-1)}+S^{(k-1)}|}{2\tau}+\frac{\kappa+2-\tau^2}{2F}\cdot|R^{(k)}-S^{(k)}-R^{(k-1)}+S^{(k-1)}|\cdot\tau\right. \\
&\qquad\left.+\frac{\kappa+2-2\tau^2}{F}\cdot|R^{(k)}-R^{(k-1)}|\cdot\tau\right\}(\tau,z_-(\tau))\,\mathrm{d}\tau \\
&\le \int_0^{\xi}\left\{\frac{M}{2}\tau\left(\frac{2}{3}\right)^{k-1}+M\tau^3\left(\frac{2}{3}\right)^{k-1}+2M\tau^3\left(\frac{2}{3}\right)^{k-1}\right\}\mathrm{d}\tau\le M\xi^2\left(\frac{2}{3}\right)^{k}.
\end{aligned}
\tag{3.42}
$$

For the term $|R^{(k+1)}-S^{(k+1)}-R^{(k)}+S^{(k)}|$, we divide it into two terms

$$
\left|R^{(k+1)}(\xi,\eta)-S^{(k+1)}(\xi,\eta)-R^{(k)}(\xi,\eta)+S^{(k)}(\xi,\eta)\right|\le I_3+I_4,
\tag{3.43}
$$

where

$$
\begin{aligned}
I_3=\int_{\xi_1}^{\xi}\Bigg\{&\frac{|R^{(k)}-S^{(k)}-R^{(k-1)}+S^{(k-1)}|}{2\tau}(\tau,z_+(\tau)) \\
&+\frac{|R^{(k)}-S^{(k)}-R^{(k-1)}+S^{(k-1)}|}{2\tau}(\tau,z_-(\tau)) \\
&+\frac{\kappa+2-\tau^2}{2F}|R^{(k)}-S^{(k)}-R^{(k-1)}+S^{(k-1)}|(\tau,z_+(\tau))\cdot\tau \\
&+\frac{\kappa+2-\tau^2}{2F}|R^{(k)}-S^{(k)}-R^{(k-1)}+S^{(k-1)}|(\tau,z_-(\tau))\cdot\tau \\
&+\frac{\kappa+2-2\tau^2}{F}\Big[|R^{(k)}-R^{(k-1)}|(\tau,z_+(\tau))+|S^{(k)}-S^{(k-1)}|(\tau,z_-(\tau))\Big]\tau\Bigg\}\,\mathrm{d}\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
I_4=\int_0^{\xi_1}\Bigg\{&\frac{|S^{(k)}-R^{(k)}-S^{(k-1)}+R^{(k-1)}|}{2\tau}+\frac{\kappa+2-\tau^2}{2F}\cdot|R^{(k)}-S^{(k)}-R^{(k-1)}+S^{(k-1)}|\tau \\
&+\frac{\kappa+2-2\tau^2}{F}\cdot|S^{(k)}-S^{(k-1)}|\cdot\tau\Bigg\}(\tau,z_-(\tau))\,\mathrm{d}\tau.
\end{aligned}
$$

By the induction assumptions, one gets

$$
\begin{aligned}
I_3 &\le \int_{\xi_1}^{\xi} \left\{ M\tau \left(\frac{2}{3}\right)^{k-1} + 6M\tau^3 \left(\frac{2}{3}\right)^{k-1} \right\} \mathrm{d}\tau \\
&\le \frac{M}{2}(\xi^2 - \xi_1^2)\left(\frac{2}{3}\right)^{k-1} + \frac{3M}{2}(\xi^4 - \xi_1^4)\left(\frac{2}{3}\right)^{k-1},
\end{aligned} \tag{3.44}
$$

and

$$
I_4 \le \int_0^{\xi_1} \left\{ \frac{M}{2}\tau \left(\frac{2}{3}\right)^{k-1} + 3M\tau^3 \left(\frac{2}{3}\right)^{k-1} \right\} \mathrm{d}\tau \le \frac{M}{4}\xi_1^2 \left(\frac{2}{3}\right)^{k-1} + \frac{3M}{4}\xi_1^4 \left(\frac{2}{3}\right)^{k-1}. \tag{3.45}
$$

We put (3.44) and (3.45) into (3.43) and recall $\delta < \frac{1}{4}$ to arrive at

$$
\begin{aligned}
&\left| R^{(k+1)}(\xi,\eta) - S^{(k+1)}(\xi,\eta) - R^{(k)}(\xi,\eta) + S^{(k)}(\xi,\eta) \right| \\
\le &\left\{ \frac{M}{2}(\xi^2 - \xi_1^2)\left(\frac{2}{3}\right)^{k-1} + \frac{3M}{2}(\xi^4 - \xi_1^4)\left(\frac{2}{3}\right)^{k-1} \right\} \\
&+ \left\{ \frac{M}{4}\xi_1^2 \left(\frac{2}{3}\right)^{k-1} + \frac{3M}{4}\xi_1^4 \left(\frac{2}{3}\right)^{k-1} \right\} \\
\le &M\xi^2 (\frac{1}{2} + \frac{3}{2}\delta^2)\left(\frac{2}{3}\right)^{k-1} \le M\xi^2 \left(\frac{2}{3}\right)^{k}.
\end{aligned} \tag{3.46}
$$

Combining with (3.41)-(3.42) and (3.46) completes the proof of the lemma. □

3.2.3. The existence and uniqueness of solutions

We now analyze the properties of the sequences $(R^{(k)}, S^{(k)})(\xi,\eta)$ defined by (3.23). From Lemma 3.3, we know that the sequences $(R^{(k)}, S^{(k)})(\xi,\eta)$ converge uniformly. It is obvious that the limit functions of the sequences $(R^{(k)}, S^{(k)})$, denoted by (R, S), are continuous. Thanks to (3.29), the functions (R, S) satisfy

$$
|R(\xi,\eta)| \le 3M\xi^2, \quad |S(\xi,\eta)| \le 3M\xi^2, \quad |R(\xi,\eta) - S(\xi,\eta)| \le 3M\xi^2 \tag{3.47}
$$

for any $(\xi,\eta) \in \Omega$. Moreover, it is easy to see that the limit functions (R, S) satisfy the system of integral equations (3.21) and the boundary conditions $R(0,\eta) = S(0,\eta) = 0$. In addition, according to (3.22), we find that $\eta = 0$ if and only if $\xi = \xi_1$, from which and (3.21) we have $R(\xi, 0) = b_1(\xi)$.

We next check the boundary conditions $R_\xi(0,\eta) = S_\xi(0,\eta) = 0$. To this end, we first claim that (R, S) are C^1 functions. Due to (3.21) and (3.47), it is clear that $R(\xi,\eta)$ and $S(\xi,\eta)$ possess one continuous derivative with respect to ξ. To establish the existence of (R_η, S_η) near $\xi = 0$, we consider the case $\eta \ge \bar{z}(\xi)$ and differentiate (3.23) with respect to η to obtain the following system of integral equations

$$\begin{cases} S_\eta^{(k)}(\xi,\eta) = \displaystyle\int_0^\xi \Big\{ \frac{S_z^{(k-1)} - R_z^{(k-1)}}{2\tau} - \frac{\kappa+2-\tau^2}{2F}(R_z^{(k-1)} - S_z^{(k-1)})\tau \\ \qquad\qquad - \frac{\kappa+2-2\tau^2}{F} S_z^{(k-1)}\tau + F_{2z}\tau \Big\} \cdot \frac{\partial z_-}{\partial \eta}(\tau, z_-(\tau))\,\mathrm{d}\tau, \\ R_\eta^{(k)}(\xi,\eta) = \displaystyle\int_0^\xi \Big\{ \frac{R_z^{(k-1)} - S_z^{(k-1)}}{2\tau} + \frac{\kappa+2-\tau^2}{2F}(R_z^{(k-1)} - S_z^{(k-1)})\tau \\ \qquad\qquad - \frac{\kappa+2-2\tau^2}{F} R_z^{(k-1)}\tau + F_{1z}\tau \Big\} \cdot \frac{\partial z_+}{\partial \eta}(\tau, z_+(\tau))\,\mathrm{d}\tau, \end{cases} \tag{3.48}$$

where

$$\frac{\partial z_\pm}{\partial \eta}(\tau;\xi,\eta) = \exp\Big(\int_\xi^\tau \frac{\partial z_\pm}{\partial z}(s, z_\pm(s;\xi,\eta))\,\mathrm{d}s \Big). \tag{3.49}$$

By the expressions of $\lambda_\pm$ in (3.20), we know that $\frac{\partial z_\pm}{\partial z} \equiv 0$ and then $\frac{\partial z_\pm}{\partial \eta} = 1$. Recalling the expressions of F_1, F_2 and the regularities of a_0, a_1, we also have

$$\begin{aligned} &|F_{1z}(\tau, z_+(\tau;\xi,\eta)) - F_{2z}(\tau, z_-(\tau;\xi,\eta))| \\ \le& \frac{2\sqrt{1-\tau^2}}{F}(|a_0''| + |a_1''|(\tau)\tau + \frac{\kappa+2-2\tau^2}{F}|a_0'(z_+(\tau;\xi,\eta)) - a_0'(z_-(\tau;\xi,\eta))| \\ &+ \frac{|a_1'(z_+(\tau;\xi,\eta))| + |a_1'(z_-(\tau;\xi,\eta))|}{F}\tau^3 \\ \le& 3M\xi. \end{aligned} \tag{3.50}$$

The last inequality holds by the derivation as in (3.32). We perform the same iteration process employed in solving (3.23) to obtain that the sequences $(R_\eta^{(k)}, S_\eta^{(k)})(\xi,\eta)$ converge uniformly, which implies that the functions $(R_\eta, S_\eta)(\xi,\eta)$ are continuous. Moreover, we have by (3.48) that $R_\eta(0,\eta) = S_\eta(0,\eta) = 0$. Since the functions (R, S) satisfy (3.21) and has the required differentiability properties, it is the solution of (3.19) satisfying boundary conditions (3.18).

For establishing the uniqueness of the solution, we consider the difference of solutions. Let (R_1, S_1) and (R_2, S_2) be two solutions of (3.19). Denote $\widetilde{R} = R_2 - R_1$ and $\widetilde{S} = S_2 - S_1$. Then $(\widetilde{R}, \widetilde{S})$ satisfy the homogeneous integral system

$$\begin{aligned} \widetilde{S}(\xi,\eta) &= \int_0^\xi \Big\{ \frac{\widetilde{S} - \widetilde{R}}{2\tau} - \frac{\kappa+2-\tau^2}{2F}(\widetilde{R} - \widetilde{S})\tau - \frac{\kappa+2-2\tau^2}{F}\widetilde{S}\tau \Big\}\,\mathrm{d}\tau, \\ \widetilde{R}(\xi,\eta) &= \begin{cases} \displaystyle\int_0^\xi \Big\{ \frac{\widetilde{R} - \widetilde{S}}{2\tau} + \frac{\kappa+2-\tau^2}{2F}(\widetilde{R} - \widetilde{S})\tau - \frac{\kappa+2-2\tau^2}{F}\widetilde{R}\tau \Big\}\,\mathrm{d}\tau, & \eta \ge \bar{z}(\xi), \\ \displaystyle\int_{\xi_1}^\xi \Big\{ \frac{\widetilde{R} - \widetilde{S}}{2\tau} + \frac{\kappa+2-\tau^2}{2F}(\widetilde{R} - \widetilde{S})\tau - \frac{\kappa+2-2\tau^2}{F}\widetilde{R}\tau \Big\}\,\mathrm{d}\tau, & \eta < \bar{z}(\xi). \end{cases} \end{aligned} \tag{3.51}$$

It is noted that the functions $(\widetilde{R}, \widetilde{S})$ also satisfy the inequalities (3.40). We repeat the insertion of these in the right side of (3.51) to obtain that $(\widetilde{R}, \widetilde{S})$ must satisfy the inequalities of the forms $|\widetilde{R}| \le \widetilde{M}(\frac{2}{3})^k$ and $|\widetilde{S}| \le \widetilde{M}(\frac{2}{3})^k$ for arbitrary k, where $\widetilde{M}$ is a positive constant. Thus we have $\widetilde{R} = \widetilde{S} \equiv 0$.

Finally, we note by (3.11), (3.13) and (3.17) that the initial-boundary value problem (3.19) (3.18) is equivalent to the initial-boundary value problem (3.4) (3.9). Hence the proof of Theorem 3.1 is complete.

4. Solutions in the physical plane

In this section, we convert the solution $(U(t,r), V(t,r))$ in the partial hodograph plane to that in the original physical plane.

According to the coordinate transformation (3.1), we have

$$\frac{\partial x}{\partial t} = \frac{\theta_y}{J}, \quad \frac{\partial y}{\partial t} = -\frac{\theta_x}{J}, \quad \frac{\partial x}{\partial r} = \frac{\sin\omega\omega_y}{J}, \quad \frac{\partial y}{\partial r} = -\frac{\sin\omega\omega_x}{J}, \tag{4.1}$$

where J is the Jacobian of the transformation defined in (3.2). In addition, it follows by (2.8), (2.13) and (2.14) that

$$\begin{aligned} \theta_x &= (t\sin r - \sqrt{1-t^2}\cos r)U + (t\sin r + \sqrt{1-t^2}\cos r)V, \\ \theta_y &= -(t\cos r + \sqrt{1-t^2}\sin r)U - (t\cos r - \sqrt{1-t^2}\sin r)V, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \omega_x &= -\tfrac{k+1-t^2}{t^2}\left\{(t\sin r - \sqrt{1-t^2}\cos r)U - (t\sin r + \sqrt{1-t^2}\cos r)V\right\} \\ \omega_y &= \tfrac{k+1-t^2}{t^2}\left\{(t\cos r + \sqrt{1-t^2}\sin r)U - (t\cos r - \sqrt{1-t^2}\sin r)V\right\}. \end{aligned} \tag{4.3}$$

Therefore we have by (4.1)-(4.3)

$$\begin{aligned} x_t &= -\frac{(t\cos r + \sqrt{1-t^2}\sin r)U(t,r) + (t\cos r - \sqrt{1-t^2}\sin r)V(t,r)}{4F(t)U(t,r)V(t,r)}t, \\ y_t &= -\frac{(t\sin r - \sqrt{1-t^2}\cos r)U(t,r) + (t\sin r + \sqrt{1-t^2}\cos r)V(t,r)}{4F(t)U(t,r)V(t,r)}t, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} x_r &= \frac{(t\cos r + \sqrt{1-t^2}\sin r)U(t,r) - (t\cos r - \sqrt{1-t^2}\sin r)V(t,r)}{4t\sqrt{1-t^2}U(t,r)V(t,r)}, \\ y_r &= \frac{(t\sin r - \sqrt{1-t^2}\cos r)U(t,r) - (t\sin r + \sqrt{1-t^2}\cos r)V(t,r)}{4t\sqrt{1-t^2}U(t,r)V(t,r)}. \end{aligned} \tag{4.5}$$

Combining with (4.4) and (4.5) leads to

$$\frac{\mathrm{d}x(t,r_-(t))}{\mathrm{d}t} = -\frac{t\cos r + \sqrt{1-t^2}\sin r}{2F(t)V(t,r)}t, \quad \frac{\mathrm{d}y(t,r_-(t))}{\mathrm{d}t} = -\frac{t\sin r - \sqrt{1-t^2}\cos r}{2F(t)V(t,r)}t, \tag{4.6}$$

where $r_-(t)$ is the negative characteristics of (3.4) defined by

$$\frac{\mathrm{d}r_-(t)}{\mathrm{d}t} = -\lambda(t), \quad \lambda(t) = \frac{\sqrt{1-t^2}t^2}{F(t)}.$$

Denote $\Omega' := \{(t,r)|\ 0 \le t \le \delta,\ \bar{r}(t) - \delta \le r \le \bar{r}(t)\}$, which is the region Ω in the coordinates (t,r). For any point $(\hat{t},\hat{r}) \in \Omega'$, we can employ (4.6) to define the value $\hat{x} = x(\hat{t},\hat{r})$

$$x(\hat{t},\hat{r}) = \begin{cases} \hat{\theta}^{-1}(\check{r}) - \displaystyle\int_0^{\hat{t}} \frac{t\cos r_-(t;\hat{t},\hat{r}) + \sqrt{1-t^2}\sin r_-(t;\hat{t},\hat{r})}{2F(t)V(t,r_-(t;\hat{t},\hat{r}))}t\ \mathrm{d}t, & \hat{r} \le r(\hat{t}), \\ \psi^{-1}(\tilde{\theta}^{-1}(\tilde{r})) - \displaystyle\int_{\tilde{t}}^{\hat{t}} \frac{t\cos r_-(t;\hat{t},\hat{r}) + \sqrt{1-t^2}\sin r_-(t;\hat{t},\hat{r})}{2F(t)V(t,r_-(t;\hat{t},\hat{r}))}t\ \mathrm{d}t, & \hat{r} > r(\hat{t}), \end{cases} \tag{4.7}$$

where $\hat{\theta}^{-1}$, $\tilde{\theta}^{-1}$ and ψ^{-1} represent, respectively, the inverses of $\hat{\theta}$, $\tilde{\theta}$ and ψ, and

$$r_-(t;\hat{t},\hat{r}) = \hat{r} + \int_t^{\hat{t}} \lambda(t)\ \mathrm{d}t, \quad \check{r} = \hat{r} + \int_0^{\hat{t}} \lambda(t)\ \mathrm{d}t, \quad r(\hat{t}) = r_2 - \int_0^{\hat{t}} \lambda(t)\ \mathrm{d}t. \tag{4.8}$$

The numbers $\tilde{t}$ and $\tilde{r}$ in (4.7) are determined by the following equations

$$\tilde{r} = r_2 + \int_0^{\tilde{r}} \lambda(t)\ \mathrm{d}t, \quad \tilde{r} = \hat{r} + \int_{\tilde{t}}^{\hat{r}} \lambda(t)\ \mathrm{d}t. \tag{4.9}$$

We mention that the point $(\tilde{t},\tilde{r})$ is the intersection of the positive characteristic $r = r_+(t;0,r_2)$ and the negative characteristic $r = r_-(t;\hat{t},\hat{r})$. Similarly, we can also define the value $\hat{y} = y(\hat{t},\hat{r})$ by (4.6)

$$y(\hat{t},\hat{r}) = \begin{cases} \varphi(\hat{\theta}^{-1}(\check{r})) - \displaystyle\int_0^{\hat{t}} \frac{t\sin r_-(t;\hat{t},\hat{r}) - \sqrt{1-t^2}\cos r_-(t;\hat{t},\hat{r})}{2F(t)V(t,r_-(t;\hat{t},\hat{r}))}t\ \mathrm{d}t, & \hat{r} \le r(\hat{t}), \\ \tilde{\theta}^{-1}(\tilde{r}) - \displaystyle\int_{\tilde{t}}^{\hat{t}} \frac{t\sin r_-(t;\hat{t},\hat{r}) - \sqrt{1-t^2}\cos r_-(t;\hat{t},\hat{r})}{2F(t)V(t,r_-(t;\hat{t},\hat{r}))}t\ \mathrm{d}t, & \hat{r} > r(\hat{t}). \end{cases} \tag{4.10}$$

In addition, we find by using (4.1)-(4.3) that the Jacobian of the map $(t,r) \mapsto (x,r)$ is

$$j := \frac{\partial(x,y)}{\partial(t,r)} = \frac{t}{4F(t)U(t,r)V(t,r)},$$

which is strictly less than zero in $\Omega' \setminus \{t = 0\}$. This means that the map $(t, r) \mapsto (x, y)$ is an one-to-one mapping for $t \in (0, \delta]$. Thus we obtain the functions $t = t(x, y)$ and $r = r(x, y)$, and then define by (3.1)

$$\theta = r(x, y), \quad \varpi = \sin\omega = \sqrt{1 - t^2(x, y)}. \tag{4.11}$$

It is not difficult to check that the functions defined in (4.11) satisfy system (2.11), which completes the proof of Theorem 2.1. Finally, we can define the functions $(c, u, v)(x, y)$ as

$$c(x, y) = \frac{\hat{q}\sqrt{\kappa}\varpi(x, y)}{\sqrt{\kappa + \varpi^2(x, y)}}, \quad u(x, y) = c(x, y)\frac{\cos r(x, y)}{\varpi(x, y)}, \quad v(x, y) = c(x, y)\frac{\sin r(x, y)}{\varpi(x, y)}, \tag{4.12}$$

which is the classical solution of Problem 1.

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