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SONIC-SUPERSONIC SOLUTIONS TO A MIXED-TYPE BOUNDARY VALUE PROBLEM FOR THE TWO-DIMENSIONAL FULL EULER EQUATIONS*

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Abstract. We are concerned with the sonic-supersonic structures extracted from the transonic flow problems in gas dynamics. A local classical supersonic solution for the two-dimensional steady full Euler equations is established in an angular region bounded by the sonic curve and the characteristic curve. In order to overcome the challenges caused by the coupling of nonlinearity and degeneracy at the corner point, we develop a new iteration pattern to show the convergence of the iterative sequence generated by the Euler equations in terms of a partial hodograph coordinate system. The pattern developed here will be useful for studying the degenerate mixed-type boundary value problems for other related nonlinear hyperbolic systems.

Key words. full Euler equations, sonic curve, degenerate hyperbolic, classical solution, characteristic decomposition

AMS subject classifications. 35Q31, 35Q35, 35M33, 35A09, 76H05

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1. Introduction.

The transonic flow problems are one kind of the most important problems in mathematical fluid dynamics, since this kind of problem appears in various important physical situations. As early as the 1940s, Courant and Friedrichs [15] described in the famous book *Supersonic Flow and Shock Waves* that for a compressible flow passing a duct, if the Mach number is not much below one, then the flow may change to supersonic somewhere on the surface of the duct due to the convexity of the duct. It is well-known that the supersonic region is a bubble above the wall of the duct. See Figure 1 for an illustration. Similar situations occur as a flow passes over an airfoil; see, among others, the monographs of Shapiro [38], Bers [3], and Kuz'min [27].

The existence of solutions for such transonic flow problems has been extensively studied but still remain open in the "global" transonic sense mathematically; see the review paper [12]. Generally speaking, a transonic structure includes the subsonic and supersonic parts which are separated by a curve composed by sonic curves or transonic shocks. Due to the nonlinearity of the governing system, the separated curve in general is a free boundary which is determined together with the solutions. Moreover, the governing system may change types around a sonic curve and is degenerate on the sonic curve. These features cause the transonic flow problems generally more difficult than the study of problems in purely subsonic or purely supersonic regions.

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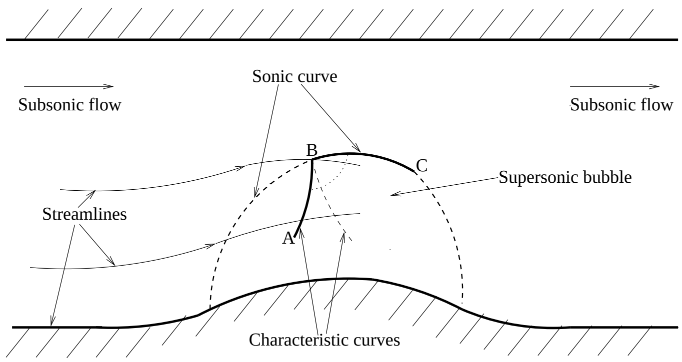

Fig. 1. *Transonic phenomena in a duct.*

There has been a great amount of effort and discussion on the transonic flow problem described in Figure 1 since the 1950s. Some explicit examples of transonic flows were presented in [3, 38, 27] and the existence conditions of continuous sonic-supersonic flows were given in [2]. In [34, 35], Morawetz indicated that a smooth transonic flow does not exist in general, which means there may exist a transonic shock in the downstream flow. The existence of weak solutions in the compensated-compactness framework was studied in [36, 5, 9]. For the subsonic side, the existence of global subsonic-sonic solutions have been intensively investigated in the past few years. Xie and Xin [43] verified the existence of global solutions in a subsonic-sonic part of the nozzle. The well-posedness for the subsonic and subsonic-sonic flows with critical mass flux was solved in [44] for the isentropic case and in [6] for the full Euler equations. In [8], Chen, Huang, and Wang discussed the subsonic-sonic limit of approximate solutions for the multidimensional full Euler equations. The properties of sonic curves were discussed for the two-dimensional (2-D) steady smooth subsonic-sonic and transonic potential flows in [41]. In a recent paper, Wang and Xin [42] established the existence and uniqueness of smooth transonic flows of Meyer type in Laval nozzles for the potential equation. We refer the reader to [40, 4, 16, 17, 18] for more related results about the subsonic-sonic solutions and to [7, 10, 11, 13, 20, 45] and references therein for the study of transonic shocks arising in supersonic flow past a blunt body. It is worthwhile to mention the work of Elling and Liu [19] about the theory of ellipticity principle for the self-similar potential flows. As for the supersonic side, the relevant results are still very limited. In [46], Zhang and Zheng constructed a local sonic-supersonic classical solution for the isentropic irrotational Euler equations. For the full Euler equations, Hu and Li studied the local existence of classical sonic-supersonic solutions in [22] and further established a global smooth supersonic-sonic solution and analyzed solution behaviors near the sonic curve in [23]. The results of the 2-D pseudosteady Euler equations can be consulted in [47, 25] for the existence of sonic-supersonic solutions and in [24, 31, 39] for the existence of semihyperbolic patch solutions.

In this paper, we are interested in the existence of classical supersonic solutions near the sonic curve in the compressible flows. Specifically, we consider a degenerate mixed-type boundary value problem in an angular region bounded by a characteristic curve and a sonic curve, which, as illustrated in Figure 1, is extracted from the transonic flow problems. One of our motivations for studying such problems arises from the Frankl problem, proposed by Frankl [21], which is to find an airfoil's arc $\widehat{DE}$ for the correctness of the transonic flow problem in the class of smooth solutions; see

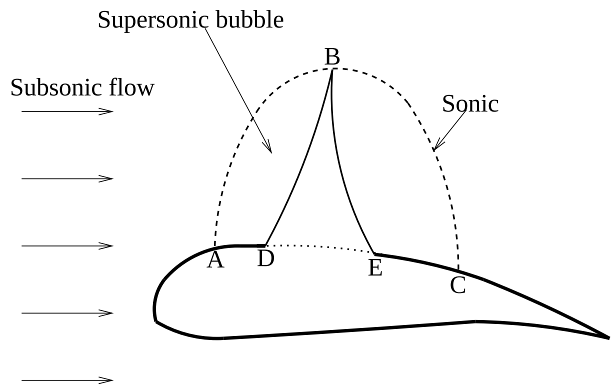

FIG. 2. *The Frankl problem: prescribed the slip condition on the arcs $\widehat{AD}$ and $\widehat{EC}$, find airfoil's arc $\widehat{DE}$ which is free of boundary conditions for the correctness of the problem in the class of smooth solutions.*

Figure 2. If the solutions in regions ADB and CEB are obtained, then one needs to solve a similar degenerate problem to determinate the arc $\widehat{DE}$. The reader may consult Morawetz [33] and Cook [14] for some uniqueness results to the Frankl problem of the linearized equation; also see Kuz'min [27] for the uniqueness discussion on a modification of the Frankl problem. In [28], Li and Hu studied such mixed-type degenerate problems for the isentropic irrotational Euler equations by introducing a set of variables to transform the Euler equations into a linear system. However, the entropy is usually not uniform in the transonic flow and the flow is not irrotational [32]. Thus it is more suitable to adopt the full system of Euler equations to characterize the transonic flow problems. We shall establish a classical sonic-supersonic solution to the degenerate mixed-type boundary value problem for the 2-D steady full Euler equations. Due to the effects of entropy and vorticity, the full Euler equations cannot be transformed into a linear system, which is much different from the isentropic irrotational case handled in [28].

The 2-D steady full Euler equations for perfect gases read that

$$\begin{cases} (\rho u)_x + (\rho v)_y = 0, \\ (\rho u^2 + p)_x + (\rho uv)_y = 0, \\ (\rho uv)_x + (\rho v^2 + p)_y = 0, \\ (\rho Eu + pu)_x + (\rho Ev + pv)_y = 0, \end{cases} \tag{1.1}$$

where ρ, (u, v), p, and E are the density, the velocity, the pressure, and the specific total energy, respectively. For polytropic gases, $E = \frac{u^2+v^2}{2} + \frac{1}{\gamma-1}\frac{p}{\rho}$, where $\gamma > 1$ is the adiabatic gas constant. The eigenvalues of system (1.1) are

$$\Lambda_0 = \Lambda_1 = \frac{v}{u}, \quad \Lambda_\pm = \frac{uv \pm c\sqrt{q^2 - c^2}}{u^2 - c^2}, \tag{1.2}$$

where $c = \sqrt{\gamma p/\rho}$ is the speed of sound and $q = \sqrt{u^2 + v^2}$ denotes the flow speed. From the expressions of $\Lambda_\pm$, it is obvious that the flow may be transonic: supersonic for $q > c$, subsonic for $q < c$, and sonic for $q = c$. The set of points on which $c = q$ is called the *sonic curve*. We consider in this paper the degenerate mixed-type problem as follows.

Problem 1. *Let $\widehat{BA}$ and $\widehat{BC}$ be two pieces of smooth curves; see Figure* 1. *We assign the boundary data on $\widehat{BA}$ and $\widehat{BC}$ such that $\widehat{BA}$ is a Λ_+-characteristic curve and $\widehat{BC}$ is a sonic curve. We look for a classical supersonic solution for* (1.1) *in the region ABC near point B.*

Since $\widehat{BA}$ is a characteristic curve and system (1.1) is degenerate on $\widehat{BC}$, Problem 1 is also called the degenerate Cauchy–Goursat problem, which is a fundamental one for the mathematical theory of transonic flow in gas dynamics. Our other motivation for discussing this kind of problems is to construct a transonic shock in the downstream region by assigning appropriate boundary data on $\widehat{BA}$ and $\widehat{BC}$ in the future. At the first glimpse, Problem 1 seems to be closely related to the degenerate initial value problem for the full Euler equations (1.1) investigated in Hu and Li [22]; one may naturally expect to solve this problem by learning from the previous approach. However, the existence framework presented in [22] cannot deal with the mixed-type boundary value problems directly. The main reason is that a key metric space adopted in [22] cannot be applied for the degenerate initial boundary value problem due to the influence of the characteristic boundary $\widehat{BA}$. Moreover, the difference between the different boundary values produced in the iteration process needs to be estimated carefully. In the current paper, we create a new fixed point iteration pattern to explore the local well-posedness of the degenerate mixed-type boundary value problems for the Euler equations. This pattern can also be applied for the other relevant transonic problems in gas dynamics, for example, the problem of transonic flow over a porous boundary [26, 27]. It is worth pointing out that, until now, there has been no general effective theory to study the degenerate mixed-type boundary value problems for the nonlinear hyperbolic problems. The framework presented here may be regarded as a meaningful trial, which is one of the most important contributions of the paper.

In order to capture the singular structures near the sonic curve, we first follow Hu and Li [22] to adopt the angle variables as the auxiliary coordinate system to transform the full Euler equations (1.1) into a new nonlinear system. This new system has an explicitly singularity-regularity structure. Based on the new system, we construct a nonlinear system of integral equations which subsequently generates an iterative sequence. The focus is on establishing the convergence of the iterative sequence. The approach in this paper is partly inspired by the work done by Berezin [1] and Protter [37] for studying the well-posedness of the Cauchy problem to the second-order linear degenerate hyperbolic equation. Compared to the linear case, there are at least two key challenges that must be addressed for the nonlinear iterative problem in our paper. First, since the eigenvalues of the system depend on the solutions, the characteristic curves also need to be iterated, which results in the integration path begin different at each iteration. To verify the convergence of the iterative sequence, we need to carefully analyze the difference of iterative fluctuations caused by the different integration paths. Second, for the mixed-type boundary value problem, the data on the characteristic boundary cannot be homogenized, which implies that the values of the iterative sequence on the boundary are different. Due to the degeneracy on the sonic curve, the difference between these different boundary values needs to be estimated at a certain order depending on the degree of degradation. The coupling of nonlinearity and singularity in the system makes the iteration process mathematically rather more difficult and complex.

The rest of the paper is organized as follows. In section 2, we formulate the problem and state the main result by introducing the angle variables and their characteristic decompositions. In section 3, we derive a new nonlinear system with clearly singularity-regularity structures in a partial hodograph plane. Moreover, the existence and uniqueness of classical solutions to the degenerate mixed-type boundary value problem for the new system are also established in this section by using the iteration method. Finally, in section 4, we return the solution in the partial hodograph plane to that in the original physical variables to complete the proof of the main result.

2. The problem in terms of angle variables and the main result. This section is devoted to formulating the degenerate mixed-type boundary value problem for the full Euler equations (1.1). In order to state the main result clearly, we introduce the Mach angles, the flow angles, the entropy, and the Bernoulli quantities as dependent variables to rewrite the governing equations. The basic characteristic decompositions in terms of these variables are provided for later restating the problem in a partial hodograph plane.

2.1. Preliminary characteristic decompositions. For smooth flows, the full Euler equations (1.1) can be written as

$$\mathbf{A}\mathbf{W}_x + \mathbf{B}\mathbf{W}_y = 0, \tag{2.1}$$

$$\mathbf{A} = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & \frac{1}{\rho} \\ 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & u \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & \frac{1}{\rho} \\ 0 & 0 & \gamma p & v \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} \rho \\ u \\ v \\ p \end{pmatrix}.$$

The eigenvalues Λ are defined by finding the roots of $\|\Lambda\mathbf{A} - \mathbf{B}\| = 0$, as expressed in (1.2). The four corresponding left eigenvectors are

$$\ell_0 = (0, u, v, 0), \quad \ell_1 = (c^2, 0, 0, -1), \quad \ell_\pm = (0, -\Lambda_\pm \gamma p, \gamma p, \Lambda_\pm u - v).$$

By a standard calculation, one obtains the characteristic form of (2.1)

$$\begin{cases} uS_x + vS_y = 0, \\ uB_x + vB_y = 0, \\ -c\rho v u_x + c\rho u v_x \pm \sqrt{u^2+v^2-c^2}\,p_x \\ \qquad +\Lambda_\pm(-c\rho v u_y + c\rho u v_y \pm \sqrt{u^2+v^2-c^2}\,p_y) = 0, \end{cases} \tag{2.2}$$

where $S = p\rho^{-\gamma}$ is the entropy function and $B = \frac{u^2+v^2}{2} + \frac{c^2}{\gamma-1}$ is the Bernoulli function.

Introduce the flow angle function θ and the Mach angle function ω as follows:

$$\tan\theta = \frac{v}{u}, \quad \sin\omega = \frac{c}{q}. \tag{2.3}$$

We denote

$$\alpha := \theta + \omega, \quad \beta := \theta - \omega. \tag{2.4}$$

One can easily check by the expressions of (1.2) that θ, α, and β are the inclination angles of characteristics, that is,

$$\tan\theta = \Lambda_0 = \Lambda_1, \quad \tan\alpha = \Lambda_+, \quad \tan\beta = \Lambda_-. \tag{2.5}$$

In addition, the speed of sound c and the velocity (u, v) can be written as functions of θ, ω, and B,

$$c=\sqrt{\frac{2\kappa\sin^2\omega}{\kappa+\sin^2\omega}B},\quad u=c\frac{\cos\theta}{\sin\omega},\quad v=c\frac{\sin\theta}{\sin\omega}, \tag{2.6}$$

where $\kappa=(\gamma-1)/2$. Furthermore, as in previous papers of Li and Zheng [29, 30], we introduce the normalized directional derivatives along the characteristics

$$\bar\partial^+=\cos\alpha\partial_x+\sin\alpha\partial_y,\quad \bar\partial^-=\cos\beta\partial_x+\sin\beta\partial_y,\quad \bar\partial^0=\cos\theta\partial_x+\sin\theta\partial_y, \tag{2.7}$$

from which one has

$$\partial_x=-\frac{\sin\beta\bar\partial^+-\sin\alpha\bar\partial^-}{\sin(2\omega)},\quad \partial_y=\frac{\cos\beta\bar\partial^+-\cos\alpha\bar\partial^-}{\sin(2\omega)},\quad \bar\partial^0=\frac{\bar\partial^++\bar\partial^-}{2\cos\omega}. \tag{2.8}$$

Then system (2.2) can be rewritten as a new system in terms of variables (S,B,θ,ω),

$$\begin{cases}\bar\partial^0S=0,\\ \bar\partial^0B=0,\\ \bar\partial^+\theta+\frac{\cos^2\omega}{\kappa+\sin^2\omega}\bar\partial^+\omega=\frac{\sin(2\omega)}{4\kappa}\left(\frac{1}{\gamma}\bar\partial^+\ln S-\bar\partial^+\ln B\right),\\ \bar\partial^-\theta-\frac{\cos^2\omega}{\kappa+\sin^2\omega}\bar\partial^-\omega=-\frac{\sin(2\omega)}{4\kappa}\left(\frac{1}{\gamma}\bar\partial^-\ln S-\bar\partial^-\ln B\right).\end{cases} \tag{2.9}$$

Denote

$$\Omega=\frac{1}{4\kappa}\left(\frac{1}{\gamma}\ln S-\ln B\right).$$

We can acquire a subsystem from (2.9),

$$\begin{cases}\bar\partial^0\Omega=0,\\ \bar\partial^+\theta+\frac{\cos\omega}{\kappa+\varpi^2}\bar\partial^+\varpi=\sin(2\omega)\bar\partial^+\Omega,\\ \bar\partial^-\theta-\frac{\cos\omega}{\kappa+\varpi^2}\bar\partial^-\varpi=-\sin(2\omega)\bar\partial^-\Omega,\end{cases} \tag{2.10}$$

where $\varpi=\sin\omega$. Hereafter the mixed variables ω and ϖ are used in a system for convenience. Once we get (Ω,θ,ϖ) from (2.10), the entropy function S and the Bernoulli function B can be obtained by solving two linear problems $\bar\partial^0S=0$ and $\bar\partial^0B=0$ with the corresponding boundary values.

We need more interpretation for the quantity Ω. Making use of (2.8) and (2.9), we find that, for any smooth function I satisfying $\bar\partial^0I=0$, we have

$$\bar\partial^0\left(\frac{\bar\partial^+I}{G(\varpi)}\right)=0,\qquad G(\varpi)=\left(\frac{\varpi^2}{\kappa+\varpi^2}\right)^{\frac{\kappa+1}{2\kappa}}.$$

The detailed derivation of the above equation can refer to Hu and Li [22]. Thus for the quantity Ω, it follows that

$$\bar\partial^0H=0,\quad H=\frac{\bar\partial^+\Omega}{G(\varpi)}. \tag{2.11}$$

Then we apply the fact $\bar\partial^-\Omega=-\bar\partial^+\Omega$ to obtain a new system in terms of the variables (H,θ,ϖ),

$$\begin{cases}\bar\partial^0H=0,\\ \bar\partial^+\theta+\frac{\cos\omega}{\kappa+\varpi^2}\bar\partial^+\varpi=\sin(2\omega)G(\varpi)H,\\ \bar\partial^-\theta-\frac{\cos\omega}{\kappa+\varpi^2}\bar\partial^-\varpi=\sin(2\omega)G(\varpi)H.\end{cases} \tag{2.12}$$

We further introduce a new variable

$$\Xi = \frac{1}{4\kappa} \ln\left(\frac{\varpi^2}{\kappa + \varpi^2}\right) - \Omega \tag{2.13}$$

to rewrite the last two equations of (2.12) as

$$\begin{cases} \bar{\partial}^+\theta + \sin(2\omega)\bar{\partial}^+\Xi = 0, \\ \bar{\partial}^-\theta - \sin(2\omega)\bar{\partial}^-\Xi = 0. \end{cases} \tag{2.14}$$

The relations between $\bar{\partial}^\pm\omega$ and $\bar{\partial}^\pm\Xi$ are

$$\bar{\partial}^\pm\omega = \frac{2\sin\omega(\kappa + \varpi^2)}{\cos\omega}[\bar{\partial}^\pm\Xi \pm HG(\varpi)]. \tag{2.15}$$

Moreover, in terms of Ξ, one has a pretty symmetrical characteristic decomposition

(2.16)

$$\begin{cases} \bar{\partial}^-\bar{\partial}^+\Xi = \frac{\kappa\bar{\partial}^+\Xi + (\kappa+\varpi^2)GH}{\cos^2\omega}[\bar{\partial}^+\Xi - \cos(2\omega)\bar{\partial}^-\Xi] + \frac{\bar{\partial}^+\Xi}{\cos^2\omega}[\bar{\partial}^+\Xi + \cos^2(2\omega)\bar{\partial}^-\Xi], \\ \bar{\partial}^+\bar{\partial}^-\Xi = \frac{\kappa\bar{\partial}^-\Xi - (\kappa+\varpi^2)GH}{\cos^2\omega}[\bar{\partial}^-\Xi - \cos(2\omega)\bar{\partial}^+\Xi] + \frac{\bar{\partial}^-\Xi}{\cos^2\omega}[\bar{\partial}^-\Xi + \cos^2(2\omega)\bar{\partial}^+\Xi]. \end{cases}$$

The above characteristic decomposition for Ξ can be gained by performing a direct calculation or consulting the previous paper Hu and Li [22]. Set $U = \bar{\partial}^+\Xi$ and $V = \bar{\partial}^-\Xi$. We arrive at the system in terms of (H, U, V) by (2.12) and (2.16),

$$\begin{cases} \bar{\partial}^0 H = 0, \\ \bar{\partial}^- U = \frac{\kappa U + (\kappa + \sin^2\omega)GH}{\cos^2\omega}[U - \cos(2\omega)V] + \frac{U}{\cos^2\omega}[U + \cos^2(2\omega)V], \\ \bar{\partial}^+ V = \frac{\kappa V - (\kappa + \sin^2\omega)GH}{\cos^2\omega}[V - \cos(2\omega)U] + \frac{V}{\cos^2\omega}[V + \cos^2(2\omega)U]. \end{cases} \tag{2.17}$$

2.2. The mixed-type boundary data and the main result. In this subsection, we set up the mixed-type boundary value problem 1 in terms of the angle variables by specifying boundary conditions on $\widehat{BC}$ and $\widehat{BA}$. Let $\widehat{BC} : y = \varphi(x)$ ($x \in [x_B, x_C]$) be a smooth curve. We assume that the curve $\widehat{BC}$ and the boundary values $(\Omega, \theta, \varpi)|_{\widehat{BC}} = (\hat{\Omega}, \hat{\theta}, \hat{\varpi})(x)$ satisfy

$$\varphi(x) \in C^3([x_B, x_C)), \quad (\hat{\Omega}, \hat{\theta})(x) \in C^3([x_B, x_C)), \quad \hat{\varpi}(x) = 1, \tag{2.18}$$

which mean that the curve $\widehat{BC}$ is a smooth sonic curve. Let $\widehat{AB} : x = \psi(y)$ ($y \in (y_A, y_B]$) be a smooth curve satisfying $x_B = \psi(y_B)$. Suppose that the curve $\widehat{AB}$ and the boundary values $(\Omega, \theta, \varpi)|_{\widehat{AB}} = (\tilde{\Omega}, \tilde{\theta}, \tilde{\varpi})(y)$ satisfy

(2.19)

$$\psi(y) \in C^1((y_A, y_B]) \cap C^4((y_A, y_B)), \quad (\tilde{\Omega}, \tilde{\theta}, \tilde{\varpi})(y) \in C^1((y_A, y_B]) \cap C^4((y_A, y_B)),$$

$$\tilde{\theta}(y) = \operatorname{arccot}\psi'(y) - \arcsin\tilde{\varpi}(y), \quad \tilde{\varpi}(y_B) = 1,$$

and

$$\begin{aligned} &\hat{\Omega}(x_B) = \tilde{\Omega}(y_B), \quad \hat{\theta}(x_B) = \tilde{\theta}(y_B), \quad \frac{\hat{\Omega}'}{\varphi'\cos\hat{\theta} - \sin\hat{\theta}}(x_B) = \frac{\tilde{\Omega}'}{\sqrt{1+(\psi')^2}}(y_B), \\ &\tilde{\theta}' + \frac{\sqrt{1-\tilde{\varpi}^2}}{\kappa+\tilde{\varpi}^2}\tilde{\varpi}' = 2\tilde{\varpi}\sqrt{1-\tilde{\varpi}^2}\tilde{\Omega}' \quad \forall\ y \in (y_A, y_B]. \end{aligned} \tag{2.20}$$

From (2.19), we see that the curve $\widehat{BA}$ is a positive characteristic and the point B is sonic. The conditions in (2.20) are the basic compatibility conditions at the point B and the curve $\widehat{BA}$. Moreover, resolving $\varpi = \tilde{\varpi}(y)$ on $\widehat{BA}$ obtains $y = \tilde{y}(\cos\tilde{\omega})$. We require the following regularity conditions to hold:

$$\begin{aligned} &\tilde{h}_0(\cos\tilde{\omega}) := \tfrac{\tilde{\Omega}'}{G(\tilde{\varpi})\sqrt{1+(\psi')^2}}(\tilde{y}(\cos\tilde{\omega})) \in C^2([0,\cos\tilde{\omega}(A))),\\ &\tilde{b}_0(\cos\tilde{\omega}) := \tfrac{1}{\sqrt{1+(\psi')^2}}\left(\tfrac{\tilde{\varpi}'}{2\tilde{\varpi}(\kappa+\tilde{\varpi}^2)} - \tilde{\Omega}'\right)(\tilde{y}(\cos\tilde{\omega})) \in C^3([0,\cos\tilde{\omega}(A))). \end{aligned} \tag{2.21}$$

The regularity conditions (2.21) imply that the functions H and U (see subsection 2.3) are, respectively, C^2- and C^3-continuous at the point B along the boundary curve $\widehat{BA}$. We comment that the derivatives of $\tilde{h}_0(y)$ and $\tilde{b}_0(y)$ may have singularities at y_B, while the functions $\tilde{h}_0(\tilde{y}(\tilde{\omega}))$ and $\tilde{b}_0(\tilde{y}(\tilde{\omega}))$ are required to be C^2 and C^3 smooth at $\cos\tilde{\omega} = 0$, respectively. These requirements can be achievable by the degeneracy of the derivative of $\tilde{y}(\cos\tilde{\omega})$.

The main result of this paper can be stated as follows.

Theorem 2.1. *Let the boundary conditions* (2.18)–(2.19) *and the basic compatibility conditions* (2.20) *and the regularity conditions* (2.21) *hold. Suppose that the higher-order compatibility conditions* ($\mathcal{C}$) *hold at the corner point (the conditions* ($\mathcal{C}$) *are given in* (3.19) *in subsection* 3.1*). Moreover, we assume that*

$$\begin{aligned} &\psi''(y_B) < 0, \quad \hat{\Omega}'(x_B) > 0, \quad \hat{\theta}'(x_B) < 0,\\ &(\varphi'\sin\hat{\theta} + \cos\hat{\theta})(x_B) > 0, \quad (\varphi'\cos\hat{\theta} - \sin\hat{\theta})(x_B) > 0. \end{aligned} \tag{2.22}$$

Then system (2.10) *with the boundary data*

$$(\Omega,\theta,\varpi)|_{\widehat{BC}} = (\hat{\Omega},\hat{\theta},\hat{\varpi})(x), \quad (\Omega,\theta,\varpi)|_{\widehat{BA}} = (\tilde{\Omega},\tilde{\theta},\tilde{\varpi})(y),$$

admits a classical supersonic solution (Ω,θ,ϖ) *in the angular region ABC around the point B.*

2.3. The boundary information for (H,U,V). For later applications, we derive in this subsection the data of (H,U,V) on the curves $\widehat{BA}$ and $\widehat{BC}$ from the boundary values of (Ω,θ,ϖ).

We first check the compatibility conditions of (2.10) at point B. Obviously, it follows from (2.18)–(2.20) that $(\hat{\Omega},\hat{\theta},\hat{\varpi})(x_B) = (\tilde{\Omega},\tilde{\theta},\tilde{\varpi})(y_B)$. Moreover, we claim that $\bar{\partial}^0\Omega = 0$ at B. In fact, along the curves $\widehat{BC}$ and $\widehat{BA}$, one has

$$\Omega_x + \varphi'\Omega_y = \hat{\Omega}', \quad \Omega_y + \psi'\Omega_x = \tilde{\Omega}',$$

from which we get

$$\Omega_x(B) = \frac{\hat{\Omega}' - \varphi'\tilde{\Omega}'}{1-\varphi'\psi'}(B), \quad \Omega_y(B) = \frac{\tilde{\Omega}' - \psi'\hat{\Omega}'}{1-\varphi'\psi'}(B).$$

Note that $\varphi'\psi' \neq 1$ at B by (2.22). Thus it follows by (2.20) that

$$\begin{aligned} &\bar{\partial}^0\Omega|_B = (\cos\theta\Omega_x + \sin\theta\Omega_y)|_B\\ &= \tfrac{(\cos\theta-\psi'\sin\theta)(\varphi'\cos\theta-\sin\theta)}{1-\varphi'\psi'}(B)\left(\tfrac{\hat{\Omega}'}{\varphi'\cos\hat{\theta}-\sin\hat{\theta}}(B) - \tfrac{\tilde{\Omega}'}{\cos\tilde{\theta}-\psi'\sin\tilde{\theta}}(B)\right)\\ &= \tfrac{(\cos\theta-\psi'\sin\theta)(\varphi'\cos\theta-\sin\theta)}{1-\varphi'\psi'}(B)\left(\tfrac{\hat{\Omega}'}{\varphi'\cos\hat{\theta}-\sin\hat{\theta}}(B) - \sin\alpha(B)\tilde{\Omega}'(B)\right)\\ &= \tfrac{(\cos\theta-\psi'\sin\theta)(\varphi'\cos\theta-\sin\theta)}{1-\varphi'\psi'}(B)\left(\tfrac{\hat{\Omega}'}{\varphi'\cos\hat{\theta}-\sin\hat{\theta}}(B) - \tfrac{\tilde{\Omega}'}{\sqrt{1+(\psi')^2}}(B)\right) = 0. \end{aligned} \tag{2.23}$$

In addition, from (2.23) and (2.8), we have $\bar{\partial}^-\Omega = -\bar{\partial}^+\Omega$ at B, which together with the fact $\cos\omega(B) = 0$ reduces the last equation of (2.10) to $\bar{\partial}^-\theta = 0$ at B by the fact. Applying (2.20) again gives $\tilde{\theta}'(B) = 0$, which implies that $\bar{\partial}^+\theta = 0$ at B. Hence

$$\begin{aligned}\bar{\partial}^-\theta(B) &= \cos(\theta(B) - \omega(B))\theta_x(B) + \sin(\theta(B) - \omega(B))\theta_y(B)\\ &= \sin\theta(B)\theta_x(B) - \cos\theta(B)\theta_y(B)\\ &= -[\cos(\theta(B) + \omega(B))\theta_x(B) + \sin(\theta(B) + \omega(B))\theta_y(B)] = -\bar{\partial}^+\theta(B) = 0,\end{aligned}$$

which means that the last equation of (2.10) holds at point B.

We now investigate the boundary data of (H, U, V). It is easy to see by (2.11) and (2.21) that

$$H|_{\widehat{BA}} = \frac{\bar{\partial}^+\Omega}{G(\varpi)}\bigg|_{\widehat{BA}} = \frac{\tilde{\Omega}'}{G(\tilde{\varpi})\sqrt{1+(\psi')^2}}(\tilde{y}(\cos\tilde{\omega})) = \tilde{h}_0(\cos\tilde{\omega}). \tag{2.24}$$

On the boundary $\widehat{BC}$, we know that

$$\Omega_x + \varphi'\Omega_y = \hat{\Omega}', \quad \cos\theta\Omega_x + \sin\theta\Omega_y = 0,$$

from which one obtains

$$\begin{aligned}\bar{\partial}^+\Omega|_{\widehat{BC}} &= (\cos\alpha\Omega_x + \sin\alpha\Omega_y)|_{\widehat{BC}} = -\sin\hat{\theta}\cdot(\Omega_x)|_{\widehat{BC}} + \cos\hat{\theta}\cdot(\Omega_y)|_{\widehat{BC}}\\ &= -\sin\hat{\theta}\bigg(-\frac{\sin\hat{\theta}\hat{\Omega}'}{\cos\hat{\theta}\varphi' - \sin\hat{\theta}}\bigg) + \cos\hat{\theta}\bigg(\frac{\cos\hat{\theta}\hat{\Omega}'}{\cos\hat{\theta}\varphi' - \sin\hat{\theta}}\bigg)\\ &= \frac{\hat{\Omega}'}{\cos\hat{\theta}\varphi' - \sin\hat{\theta}},\end{aligned}$$

which combined with (2.11) leads to

$$H|_{\widehat{BC}} = \frac{\bar{\partial}^+\Omega}{G(\varpi)}\bigg|_{\widehat{BC}} = \frac{\hat{\Omega}'}{G(1)(\cos\hat{\theta}\varphi' - \sin\hat{\theta})}(x) := \hat{h}_0(x). \tag{2.25}$$

For the data of U on $\widehat{BA}$, we use (2.15) and (2.21) to achieve

$$\begin{aligned}U|_{\widehat{BA}} &= \bigg(\frac{\bar{\partial}^+\varpi}{2\varpi(\kappa+\varpi^2)} - HG(\varpi)\bigg)\bigg|_{\widehat{BA}}\\ &= \frac{1}{\sqrt{1+(\psi')^2}}\bigg(\frac{\tilde{\varpi}'}{2\tilde{\varpi}(\kappa+\tilde{\varpi}^2)} - \tilde{\Omega}'\bigg)(\tilde{y}(\cos\tilde{\omega})) = \tilde{b}_0(\cos\tilde{\omega}).\end{aligned} \tag{2.26}$$

Furthermore, it follows from (2.8) that $\bar{\partial}^+\Xi + \bar{\partial}^-\Xi = 2\cos\omega\bar{\partial}^0\Xi$, which indicates that $\bar{\partial}^+\Xi = -\bar{\partial}^-\Xi$ on the sonic curve $\widehat{BC}$. We add the two equations in (2.14) to observe

$$\bar{\partial}^+\Xi - \bar{\partial}^-\Xi = -\frac{\bar{\partial}^+\theta + \bar{\partial}^-\theta}{\sin(2\omega)} = -\frac{\bar{\partial}^0\theta}{\sin\omega},$$

which implies that

$$U|_{\widehat{BC}} = \bar{\partial}^+\Xi|_{\widehat{BC}} = -\frac{\bar{\partial}^0\theta}{2}\bigg|_{\widehat{BC}}, \quad V|_{\widehat{BC}} = \bar{\partial}^-\Xi|_{\widehat{BC}} = \frac{\bar{\partial}^0\theta}{2}\bigg|_{\widehat{BC}}. \tag{2.27}$$

Thanks to the fact $\bar{\partial}^+\theta - \bar{\partial}^-\theta = 0$ on the sonic curve, one can find that

$$\bar{\partial}^0\theta|_{\widehat{BC}} = \frac{\hat{\theta}'}{\cos\hat{\theta} + \varphi'\sin\hat{\theta}},$$

which together with (2.27) gives

$$U|_{\widehat{BC}} = -V|_{\widehat{BC}} = -\frac{\bar{\partial}^0\theta}{2}|_{\widehat{BC}} = -\frac{\hat{\theta}'}{2(\cos\hat{\theta} + \varphi'\sin\hat{\theta})} =: -\hat{a}_0(x). \tag{2.28}$$

We next discuss the data $\bar{\partial}^0\Xi$ on the boundary $\widehat{BC}$ for later use. In view of (2.13) and the first equation of (2.10), one has

$$\bar{\partial}^0\Xi|_{\widehat{BC}} = \frac{1}{2(\kappa+1)}(\bar{\partial}^0\varpi)|_{\widehat{BC}}. \tag{2.29}$$

Adding the last two equations of (2.12) suggests

$$\bar{\partial}^0\theta + \frac{\bar{\partial}^+\varpi - \bar{\partial}^-\varpi}{2(\kappa+\varpi^2)} = 2\varpi G(\varpi)H,$$

which combined with (2.7) gets

$$-\sin\theta\varpi_x + \cos\theta\varpi_y = \frac{\kappa+\varpi^2}{\varpi}[2\varpi G(\varpi)H - \bar{\partial}^0\theta],$$

from which one has

$$-\sin\hat{\theta}\cdot(\varpi_x)|_{\widehat{BC}} + \cos\hat{\theta}\cdot(\varpi_y)|_{\widehat{BC}} = 2(\kappa+1)[G(1)\hat{h}_0 - \hat{a}_0],$$

which along with the fact $\varpi(x,\varphi(x)) \equiv 1$ yields

$$(\varpi_x)|_{\widehat{BC}} = -\frac{2(\kappa+1)[G(1)\hat{h}_0 - \hat{a}_0]\varphi'}{\cos\hat{\theta} + \varphi'\sin\hat{\theta}}, \quad (\varpi_y)|_{\widehat{BC}} = \frac{2(\kappa+1)[G(1)\hat{h}_0 - \hat{a}_0]}{\cos\hat{\theta} + \varphi'\sin\hat{\theta}}.$$

Inserting the above into (2.29), we acquire

$$\bar{\partial}^0\Xi|_{\widehat{BC}} = \frac{\sin\hat{\theta} - \varphi'\cos\hat{\theta}}{\cos\hat{\theta} + \varphi'\sin\hat{\theta}}[G(1)\hat{h}_0 - \hat{a}_0](x) := \hat{a}_1(x). \tag{2.30}$$

According to the derivation process and (2.20), we know that $\tilde{h}_0(0) = \hat{h}_0(x_B)$ and $\tilde{b}_0(0) = -\hat{a}_0(x_B)$. Moreover, it is easily confirmed by the smoothness conditions (2.18)–(2.19) that if $\varphi'\cos\hat{\theta} - \sin\hat{\theta} \neq 0$ and $\cos\hat{\theta} + \varphi'\sin\hat{\theta} \neq 0$, then the functions $\tilde{h}_0, \hat{h}_0, \hat{a}_0, \hat{a}_1$ are C^2-continuous and the function $\tilde{b}_0$ is C^3-continuous. In addition, we recall (2.22) to see that $(\varphi'\cos\hat{\theta} - \sin\hat{\theta})(x_B) > 0$, $(\cos\hat{\theta} + \varphi'\sin\hat{\theta})(x_B) > 0$, $\hat{h}_0(x_B) > 0$, $\hat{a}_0(x_B) < 0$, and $\hat{a}_1(x_B) < 0$. Furthermore, we use (2.19) and (2.20) again to arrive at

$$\left(\frac{1}{\sqrt{1+(\tilde{\varpi})^2}} - \frac{\sqrt{1-\tilde{\varpi}^2}}{\kappa+(\tilde{\varpi})^2}\right)\tilde{\varpi}' = -\frac{\psi''}{1+(\psi')^2} - 2\tilde{\varpi}\sqrt{1-\tilde{\varpi}^2}\tilde{\Omega}', \tag{2.31}$$

which along with (2.22) gives $\tilde{\varpi}'(x_B) > 0$. Thus we obtain by continuity that there exist two small constants $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $(\varphi'\cos\hat{\theta} - \sin\hat{\theta})(x) \geq \varepsilon_0$,

$(\cos\hat{\theta}+\varphi'\sin\hat{\theta})(x) > \varepsilon_0$, $\hat{h}_0(x) \geq \varepsilon_0$, $\hat{a}_0(x) \leq -\varepsilon_0$, $\hat{a}_1(x) \leq -\varepsilon_0$ for any $x \in [x_B, x_B + \delta_0]$, $\tilde{\varpi}'(y) \geq \varepsilon_0$ for any $y \in [y_B - \delta_0, y_B]$, and $\tilde{h}_0(\cos\tilde{\omega}) \geq \varepsilon_0$, $\tilde{b}_0(\cos\tilde{\omega}) \geq \varepsilon_0$ for any $\cos\tilde{\omega} \in [0, \delta_0]$. Since we only consider the existence of solutions around point B, we may assume, without loss of generality,

$$
\begin{aligned}
&(\hat{h}_0, \hat{a}_0, \hat{a}_1)(x) \in C^2([x_B, x_C)),\\
&\tilde{h}_0(\cos\tilde{\omega}) \in C^2([0, \cos\tilde{\omega}(A))), \quad \tilde{b}_0(\cos\tilde{\omega}) \in C^3([0, \cos\tilde{\omega}(A))),\\
&\hat{\theta}'(x) \leq -\varepsilon_0,\ \hat{a}_0(x) \leq -\varepsilon_0,\ \hat{h}_0(x) \geq \varepsilon_0,\ \hat{a}_1(x) \leq -\varepsilon_0\ \forall\ x \in [x_B, x_C),\\
&\tilde{\varpi}'(y) \geq \varepsilon_0\ \forall\ y \in (y_A, y_B].
\end{aligned} \tag{2.32}
$$

Otherwise, we can use the points C_1 and A_1 instead of C and A, respectively, such that the above hold on $\widehat{BC_1}$ and $\widehat{BA_1}$.

3. Solutions in a partial hodograph plane. In this section, we introduce a partial hodograph transformation to transform system (2.17) into a new system with explicitly singularity-regularity structures, and then apply the iteration method to solve the new system in an angular region corresponding to the region ABC near point B.

3.1. The problem in a partial hodograph plane. This subsection is devoted to reformulating the problem in terms of a partial hodograph coordinate system. For this end, we introduce the transformation

$$
t = \cos\omega(x, y), \quad r = \theta(x, y). \tag{3.1}
$$

By using (2.14) and (2.15), we calculate the Jacobian of the transformation (3.1)

$$
J := \frac{\partial(t, r)}{\partial(x, y)} = \sin\omega(\bar{\partial}^+\omega\bar{\partial}^-\Xi + \bar{\partial}^-\omega\bar{\partial}^+\Xi) = \frac{2F}{t}[2UV + G_1H(V-U)], \tag{3.2}
$$

where

$$
F = F(t) = (1-t^2)(\kappa+1-t^2), \quad G_1 = G_1(t) = \left(\frac{1-t^2}{\kappa+1-t^2}\right)^{\frac{\kappa+1}{2\kappa}}. \tag{3.3}
$$

Recalling the conditions in (2.32) yields $J \neq 0$ away from the boundary curve $\widehat{BA} \cup \widehat{BC}$.

In terms of the coordinates (t, r), the operators $\bar{\partial}^i (i = \pm, 0)$ can be transformed into

$$
\begin{aligned}
\bar{\partial}^+ &= -\tfrac{2F}{t}(U + G_1H)\partial_t - 2\sqrt{1-t^2}Ut\partial_r,\\
\bar{\partial}^- &= -\tfrac{2F}{t}(V - G_1H)\partial_t + 2\sqrt{1-t^2}Vt\partial_r,\\
\bar{\partial}^0 &= -\tfrac{F}{t^2}(U + V)\partial_t + \sqrt{1-t^2}(V-U)\partial_r.
\end{aligned} \tag{3.4}
$$

Therefore, we can obtain from (2.17) a new closed system for the variables (H, U, V) under the coordinates (t, r) as follows:

$$
\begin{cases}
H_t + \frac{\sqrt{1-t^2}(U-V)t^2}{F(U+V)}H_r = 0,\\
U_t - \frac{\sqrt{1-t^2}Vt^2}{F(V-G_1H)}U_r = -\frac{(\kappa+1)U+(\kappa+1-t^2)G_1H}{2F(V-G_1H)} \cdot \frac{U+V}{t} + \frac{(\kappa+2-2t^2)U+(\kappa+1-t^2)G_1H}{F(V-G_1H)}Vt,\\
V_t + \frac{\sqrt{1-t^2}Ut^2}{F(U+G_1H)}V_r = -\frac{(\kappa+1)V-(\kappa+1-t^2)G_1H}{2F(U+G_1H)} \cdot \frac{U+V}{t} + \frac{(\kappa+2-2t^2)V-(\kappa+1-t^2)G_1H}{F(U+G_1H)}Ut.
\end{cases} \tag{3.5}
$$

Once we obtain the solution $(H,U,V)(t,r)$ of (3.5), we can use them to construct the functions $x=x(t,r)$ and $y=y(t,r)$ and then establish the functions $(\theta,\varpi)(x,y)$ by (3.1).

We now derive the boundary data for system (3.5) on the (t,r) coordinates. It suggests by the assumption $\hat{\theta}'<0$ on $\widehat{BC}$ that the smooth function $r=\hat{\theta}(x)$ is strictly decreasing, from which we know that there exists an inverse function, denoted by $x=\hat{x}(r)$ $(r\in(r_1,r_2])$, where $r_1=\hat{\theta}(x_C)$ and $r_2=\hat{\theta}(x_B)$. In addition, it is easily seen that the sonic boundary $\widehat{BC}$ on the (x,y)-plane is transformed to a segment $\widehat{B'C'}$ on $t=0$ with $r\in(r_1,r_2]$ on the (t,r)-plane. Denote $\hat{h}_0(r)=\hat{h}_0(\hat{x}(r))$, $\hat{a}_0(r)=\hat{a}_0(\hat{x}(r))$, and $\hat{a}_1(r)=\hat{a}_1(\hat{x}(r))$. On the segment $\widehat{B'C'}$, we have

$$(H,U,V)(0,r)=(\hat{h}_0,-\hat{a}_0,\hat{a}_0)(r)\quad\forall\, r\in(r_1,r_2]. \tag{3.6}$$

Furthermore, if (H,U,V) is a smooth solution of (3.5), then there must be

$$H_t|_{t=0}=0,\quad U_t|_{t=0}=\frac{U+V}{2t}\bigg|_{t=0},\quad V_t|_{t=0}=\frac{U+V}{2t}\bigg|_{t=0}. \tag{3.7}$$

In view of the definitions of (U,V), we observe that the quantity $(U+V)/(2t)$ on the (t,r)-plane corresponds to the term $\bar{\partial}^0\Xi$ on the (x,y)-plane. From (2.30) and (3.7), we thus have

$$H_t(0,r)=0,\quad U_t(0,r)=\hat{a}_1(r),\quad V_t(0,r)=\hat{a}_1(r)\quad\forall\, r\in(r_1,r_2]. \tag{3.8}$$

For the boundary $\widehat{BA}$, we recall the assumption $\tilde{\varpi}'(y)\ge\varepsilon_0$ on $(y_A,y_B]$ in (2.32) to obtain by (2.19) that $\tilde{\theta}'<0$ on $\widehat{BA}\setminus\{B\}$, which means that $r=\tilde{\theta}(y)$ is a strictly decreasing function on $(y_A,y_B]$. Hence there exists an inverse function $y=\tilde{y}(r)$ on $r\in[r_2,r_3)$, where $r_3=\tilde{\theta}(y_A)$. We denote the curve $\{(t,r)|\ t=\sqrt{1-\tilde{\varpi}^2(\tilde{y}(r))}\}$ by $\widehat{B'A'}: r=\tilde{r}(t)$ $(r\in[r_2,r_3))$. Then we have the following proposition.

Proposition 1. *The curve $\widehat{B'A'}$ is the image of the curve $\widehat{BA}$ on the (t,r)-plane and is a positive characteristic of system* (3.5). *Moreover, the function $r=\tilde{r}(t)$ can be expressed as*

$$\tilde{r}(t)=r_2+\int_0^t\frac{\sqrt{1-s^2}\tilde{b}_0(s)s^2}{F(s)[\tilde{b}_0(s)+G_1(s)\tilde{h}_0(s)]}\,\mathrm{d}s,\quad t\in[0,t_0), \tag{3.9}$$

where $t_0=\sqrt{1-\tilde{\varpi}^2(y_A)}$.

Proof. It suffices to show that the image of $\widehat{BA}$ on the (t,r)-plane and the curve $\widehat{B'A'}$ both are positive characteristic curves of system (3.5) passing point $(0,r_2)$. Differentiating the equality $x(t,r)=\psi(y(t,r))$ with respect to t and applying the fact $\psi'=\cot(\theta+\omega)$ gains

$$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\cot(\theta+\omega)y_t-x_t}{x_r-\cot(\theta+\omega)y_r},$$

which together with (3.1) and (2.14)–(2.15) acquires

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}t}&=-\frac{\cot(\theta+\omega)\theta_x+\theta_y}{\sin\omega(\omega_y+\cot(\theta+\omega)\omega_x)}\\&=-\frac{\bar{\partial}^+\theta}{\sin\omega\bar{\partial}^+\omega}=\frac{\cos^2\omega\bar{\partial}^+\Xi}{\sin\omega(\kappa+\sin^2\omega)(\bar{\partial}^+\Xi+G_1H)}=\frac{\sqrt{1-t^2}U}{F(t)(U+G_1H)}t^2,\end{aligned} \tag{3.10}$$

which indicates that the image of $\widehat{BA}$ on the (t, r)-plane is a positive characteristic of system (3.5). Furthermore, it is easy to see that $x(0, r_2) = x_B$ and $y(0, r_2) = y_B$. Thus the curve defined by (3.10) passes through point $(0, r_2)$.

For the curve $\widehat{B'A'}$, we differentiate the equality $t = \sqrt{1 - \tilde{\varpi}^2(\tilde{y}(r))}$ with respect to t to suggest

$$\frac{\mathrm{d}\tilde{r}}{\mathrm{d}t} = -\frac{\sqrt{1 - \tilde{\varpi}^2(\tilde{y}(r))}}{\tilde{\varpi}\tilde{\varpi}'} \cdot \tilde{\theta}'(y),$$

which along with (2.20) and (2.26) leads to

$$\begin{aligned}\frac{\mathrm{d}\tilde{r}}{\mathrm{d}t} &= \frac{2(1 - \tilde{\varpi}^2(\tilde{y}(r)))}{\tilde{\varpi}'}\left(\frac{\tilde{\varpi}'}{2\tilde{\varpi}(\kappa + \tilde{\varpi}^2)} - \tilde{\Omega}'\right)\\ &= \frac{2t^2}{\tilde{\varpi}'}\tilde{b}_0\sqrt{1 + (\psi')^2} = \frac{2t^2\tilde{b}_0}{2\tilde{\varpi}(\kappa + \tilde{\varpi}^2)(\tilde{b}_0 + G_1\tilde{h}_0)} = \frac{\sqrt{1 - t^2}\tilde{b}_0}{F(t)(\tilde{b}_0 + G_1\tilde{h}_0)}t^2,\end{aligned} \tag{3.11}$$

from which we find that $\widehat{B'A'}$ is a positive characteristic of system (3.5). It is obvious that $\tilde{r}(0) = r_2$. The expression (3.9) follows directly from (3.11) and the proof of the proposition is complete. ∎

Let function $\tilde{s}_0(t)$ be the solution of the following ODE problem:

$$\begin{cases} \frac{\mathrm{d}\tilde{s}_0(t)}{\mathrm{d}t} = -\frac{(\kappa+1)\tilde{s}_0 - (\kappa+1-t^2)G_1\tilde{h}_0}{2F(\tilde{b}_0 + G_1\tilde{h}_0)} \cdot \frac{\tilde{b}_0 + \tilde{s}_0}{t} + \frac{(\kappa+2-2t^2)\tilde{s}_0 - (\kappa+1-t^2)G_1\tilde{h}_0}{F(\tilde{b}_0 + G_1\tilde{h}_0)}\tilde{b}_0 t,\\ \tilde{s}_0(0) = \hat{a}_0(r_2). \end{cases} \tag{3.12}$$

The solvability of problem (3.12) will be shown in Lemma 3.1 in subsection 3.2.1. Therefore, on the curve $\widehat{B'A'}$, we have

$$(H, U, V)(t, \tilde{r}(t)) = (\tilde{h}_0, \tilde{b}_0, \tilde{s}_0)(t) \quad \forall\, t \in [0, t_0). \tag{3.13}$$

Summing up (3.6), (3.13), and (3.8), we finally arrive at the mixed-type boundary data of system (3.5) as follows:

$$\begin{aligned}&(H, U, V)(0, r) = (\hat{h}_0, -\hat{a}_0, \hat{a}_0)(r),\\ &(H_t, U_t, V_t)(0, r) = (0, \hat{a}_1, \hat{a}_1)(r) \ \ \forall\, r \in (r_1, r_2],\\ &(H, U, V)(t, \tilde{r}(t)) = (\tilde{h}_0, \tilde{b}_0, \tilde{s}_0)(t) \ \ \forall\, t \in [0, t_0).\end{aligned} \tag{3.14}$$

According to the above derivation process and (2.32), we know that the functions $\hat{h}_0, \tilde{h}_0, \hat{a}_0$, $\hat{a}_1$, and $\tilde{b}_0$ satisfy

$$\begin{aligned}&(\hat{h}_0, \hat{a}_0, \hat{a}_1) \in C^2((r_1, r_2]),\ \tilde{h}_0 \in C^2([0, t_0)),\ \tilde{b}_0 \in C^3([0, t_0)),\\ &\hat{h}_0 \geq \varepsilon_0,\ \hat{a}_0 \leq -\varepsilon_0,\ \hat{a}_1 \leq -\varepsilon_0,\\ &\tilde{h}_0(0) = \hat{h}_0(r_2),\ \tilde{h}_0'(0) = 0,\ \tilde{b}_0(0) = -\hat{a}_0(r_2),\ \tilde{b}_0'(0) = \hat{a}_1(r_2).\end{aligned} \tag{3.15}$$

We comment that the conditions in the last line of (3.15) are the basic compatibility conditions at the point $B'(0, r_2)$, which follow from the definitions of functions $(\hat{h}_0, \tilde{h}_0, \hat{a}_0, \hat{a}_1, \tilde{b}_0)$ and the compatibility conditions (2.20). Hence, the problem in terms of (t, r) coordinates can be restated

PROBLEM 2. *Under the assumption* (3.15)*, we seek a local classical solution for system* (3.5) *with mixed-type boundary conditions* (3.14) *in the region* $t > 0$ *near the point* $B'(0, r_2)$.

To solve Problem 2, we need some information of H_r and U_r on the boundary $\widehat{B'A'}$. According to the equation of H in (3.5) and the boundary value $H(t,\tilde{r}(t)) = \tilde{h}_0(t)$, one obtains

$$
\begin{cases} H_t(t,\tilde{r}(t)) + \dfrac{\sqrt{1-t^2}(\tilde{b}_0 - \tilde{s}_0)t^2}{F(\tilde{b}_0+\tilde{s}_0)} H_r(t,\tilde{r}(t)) = 0, \\ H_t(t,\tilde{r}(t)) + \dfrac{\sqrt{1-t^2}\tilde{b}_0 t^2}{F[\tilde{b}_0 + G_1\tilde{h}_0]} H_r(t,\tilde{r}(t)) = \tilde{h}_0'(t), \end{cases} \tag{3.16}
$$

from which we get

$$
H_r(t,\tilde{r}(t)) = \frac{F(\tilde{b}_0+\tilde{s}_0)}{t\sqrt{1-t^2}[(\tilde{s}_0-\tilde{b}_0)(\tilde{b}_0+G_1\tilde{h}_0)+\tilde{b}_0(\tilde{b}_0+\tilde{s}_0)]} \cdot \frac{\tilde{h}_0'(t)}{t} := \tilde{g}_1(t). \tag{3.17}
$$

A similar argument for U achieves

$$
U_r(t,\tilde{r}(t)) = \frac{F(\tilde{b}_0+G_1\tilde{h}_0)(\tilde{s}_0-G_1\tilde{h}_0)}{\sqrt{1-t^2}[2\tilde{b}_0\tilde{s}_0+G_1\tilde{h}_0(\tilde{s}_0-\tilde{b}_0)]} \cdot \frac{\tilde{b}_0'(t)-g(t)}{t^2} := \tilde{g}_2(t), \tag{3.18}
$$

where

$$
\begin{aligned} g(t) \\ = -\frac{(\kappa+1)\tilde{b}_0+(\kappa+1-t^2)G_1\tilde{h}_0}{2F(\tilde{s}_0-G_1\tilde{h}_0)} \cdot \frac{\tilde{b}_0+\tilde{s}_0}{t} + \frac{(\kappa+2-2t^2)\tilde{b}_0+(\kappa+1-t^2)G_1\tilde{h}_0}{F(\tilde{s}_0-G_1\tilde{h}_0)}\tilde{s}_0 t. \end{aligned}
$$

Actually, $g(t)$ is the value of the right-hand term of the equation for U in (3.5) at boundary $\widehat{B'A'}$. Now we further assume that the functions $\tilde{h}_0(t)$ and $\tilde{b}_0(t)$ satisfy some appropriate conditions such that the following compatibility conditions $(\mathcal{C})$ hold:

$$
(\mathcal{C}): \begin{cases} (\mathcal{C}_1): \ \tilde{g}_1(0) = \hat{h}_0'(r_2), \quad \tilde{g}_2(0) = -\hat{a}_0'(r_2), \\ (\mathcal{C}_2): \qquad\qquad\qquad\qquad\quad \tilde{g}_2'(0) = \hat{a}_1'(r_2). \end{cases} \tag{3.19}
$$

Remark 1. The conditions $(\mathcal{C})$ are the compatibility conditions of H_r and U_r along the characteristic boundary $\widehat{B'A'}$ and the line $t=0$ at the point B'. More precisely, the condition $(\mathcal{C}_1)$ ensures that H_r and U_r are continuous at corner B', which is reasonable when considering the existence of classical solutions. The condition $(\mathcal{C}_2)$ comes from the continuity of the derivative of U_r with respect to t, which plays a key role in dealing with the singularities in the current paper and can be properly relaxed; see Remark 2 in subsection 3.2.3.

Therefore, we have the the following existence theorem.

THEOREM 3.1. *Suppose that* (3.15) *and* (3.19) *hold. The mixed-type boundary value problem* (3.5), (3.14) *admits a unique classical solution around point* $B'(0,r_2)$.

3.2. The proof of Theorem 3.1. This subsection serves to solve the mixed-type boundary value problem (3.5), (3.14). The proof is divided into four steps. First, we introduce new variables to homogenize the boundary conditions and then give the definition of admissible functions. Second, we derive the integral equations and then construct an iterative sequence. Third, we establish several key lemmas for the iterative sequence. Finally, we complete the proof of Theorem 3.1.

3.2.1. The homogeneous problem. For convenience, we introduce $z = r_2 - r$. Then problem (3.5), (3.14) can be rewritten as

$$\text{(3.20)}\quad \begin{cases} \overline{H}_t - \frac{\sqrt{1-t^2}(\overline{U}+\overline{V})t^2}{F(\overline{U}-\overline{V})}\overline{H}_z = 0, \\ \overline{U}_t + \frac{\sqrt{1-t^2}\overline{V}t^2}{F(\overline{V}+G_1\overline{H})}\overline{U}_z = \frac{(\kappa+1)\overline{U}+(\kappa+1-t^2)G_1\overline{H}}{2F(\overline{V}+G_1\overline{H})} \cdot \frac{\overline{U}-\overline{V}}{t} + \frac{(\kappa+2-2t^2)\overline{U}+(\kappa+1-t^2)G_1\overline{H}}{F(\overline{V}+G_1\overline{H})}\overline{V}t, \\ \overline{V}_t - \frac{\sqrt{1-t^2}\overline{U}t^2}{F(\overline{U}+G_1\overline{H})}\overline{V}_z = \frac{(\kappa+1)\overline{V}+(\kappa+1-t^2)G_1\overline{H}}{2F(\overline{U}+G_1\overline{H})} \cdot \frac{\overline{V}-\overline{U}}{t} + \frac{(\kappa+2-2t^2)\overline{V}+(\kappa+1-t^2)G_1\overline{H}}{F(\overline{U}+G_1\overline{H})}\overline{U}t, \end{cases}$$

with

$$\text{(3.21)}\quad \begin{array}{l} (\overline{H},\overline{U},\overline{V})(0,z) = (h_0, a_0, a_0)(z), \\ (\overline{H}_t,\overline{U}_t,\overline{V}_t)(0,z) = (0,-a_1,a_1)(z) \quad \forall\, z \in [0, r_2 - r_1), \\ (\overline{H},\overline{U},\overline{V})(t,\tilde{z}(t)) = (\tilde{h}_0, \tilde{b}_0, -\tilde{s}_0)(t) \quad \forall\, t \in [0, t_0), \end{array}$$

where $(\overline{H},\overline{U},\overline{V})(t,z) = (H,U,-V)(t, r_2 - z)$, $(h_0,a_0,a_1)(z) = (\hat{h}_0, -\hat{a}_0, -\hat{a}_1)(r_2 - z)$, and $\tilde{z}(t) = r_2 - \tilde{r}(t)$.

We further introduce the following variables (W, R, S) to homogenize the boundary conditions (3.21):

$$\text{(3.22)}\quad W = \overline{H} - h_0(z),\ R = \overline{U} - a_0(z) + a_1(z)t,\ S = \overline{V} - a_0(z) - a_1(z)t.$$

Combining with (3.21) and (3.22) gives

$$\text{(3.23)}\quad \begin{array}{l} (W,R,S,W_t,R_t,S_t)(0,z) = 0 \quad \forall\, z \in [0, r_2 - r_1), \\ (W,R,S)(t,\tilde{z}(t)) = (h,b,s)(t) \quad \forall\, t \in [0,t_0), \end{array}$$

where

$$h(t) = \tilde{h}_0(t) - h_0(\tilde{z}(t)),$$
$$b(t) = \tilde{b}_0(t) - a_0(\tilde{z}(t)) + a_1(\tilde{z}(t))t, \quad s(t) = -\tilde{s}_0(t) - a_0(\tilde{z}(t)) - a_1(\tilde{z}(t))t.$$

We recall (3.15) to get

$$\text{(3.24)}\quad \begin{array}{l} (h_0,a_0,a_1)(z) \in C^2([0, r_2 - r_1)),\ h(t) \in C^2([0,t_0)),\ b(t) \in C^3([0,t_0)), \\ h_0 \geq \varepsilon_0, \quad a_0 \geq \varepsilon_0, \quad a_1 \geq \varepsilon_0, \\ h(0) = h'(0) = b(0) = b'(0) = 0. \end{array}$$

In addition, one uses (3.17)–(3.18) and (3.22) to acquire

$$\text{(3.25)}\quad \begin{array}{r} W_z(t,\tilde{z}(t)) = -\tilde{g}_1(t) - h_0'(\tilde{z}(t)) := g_1(t), \\ R_z(t,\tilde{z}(t)) = -\tilde{g}_2(t) - a_0'(\tilde{z}(t)) + a_1'(\tilde{z}(t))t := g_2(t), \end{array}$$

which together with the compatibility conditions (3.19), the regularity conditions in (3.15), and the facts $h_0' = -\hat{h}_0'$, $a_0' = \hat{a}_0'$, $a_1' = \hat{a}_1'$ arrive at

$$\text{(3.26)}\quad |g_1(t)| \leq \widehat{K}t, \quad |g_2(t)| \leq \widehat{K}t^2$$

for some positive constant $\widehat{K}$.

By performing direct but tedious and lengthy calculations and doing some arrangements, we can obtain the system in terms of (W, R, S),

$$
(3.27)\qquad \begin{cases}
W_t + \dfrac{\sqrt{1-t^2}(R+S+2a_0)t^2}{F[2a_1t-(R-S)]}W_z = C_1 t,\\
R_t + \dfrac{\sqrt{1-t^2}(S+a_0+a_1t)t^2}{F(S+G_1W+\Psi)}R_z = \dfrac{R-S}{2t} + A_1\dfrac{(R-S)^2}{t}\\
\qquad\qquad + A_2R + A_3S + A_4tW + A_5t^2 + A_6t,\\
S_t - \dfrac{\sqrt{1-t^2}(R+a_0-a_1t)t^2}{F(R+G_1W+\Phi)}S_z = \dfrac{S-R}{2t} + B_1\dfrac{(R-S)^2}{t}\\
\qquad\qquad + B_2R + B_3S + B_4tW + B_5t^2 + B_6t,
\end{cases}
$$

where $\Psi = a_0(z) + a_1(z)t + G_1(t)h_0(z), \Phi = a_0(z) - a_1(z)t + G_1(t)h_0(z)$,

$$
C_1 = -\frac{\sqrt{1-t^2}(R+S+2a_0)}{F[2a_1t-(R-S)]}h_0't, \quad A_1 = \frac{\kappa+1}{2F(S+G_1W+\Psi)},
$$

$$
A_2 = -\frac{4(\kappa+1)a_1 + G_1(W+h_0)t}{2F(S+G_1W+\Psi)} + \frac{(\kappa+2-t^2)t}{2F} + \frac{(\kappa+2-2t^2)(S+a_0+a_1t)t}{F(S+G_1W+\Psi)},
$$

$$
\begin{aligned}
A_3 = {} & \frac{2(\kappa+1)a_1}{F(S+G_1W+\Psi)} + \frac{(\kappa+2-3t^2)t}{2F} - \frac{t(1-2t^2)G_1(W+h_0)}{2F(S+G_1W+\Psi)}\\
& - \frac{t(\kappa+2-2t^2)(S+a_0+a_1t)}{F(S+G_1W+\Psi)} + \frac{t(G_1h_0a_0 - 2(\kappa+1)a_1^2)}{F\Psi(S+G_1W+\Psi)},
\end{aligned}
$$

$$
A_4 = -\frac{G_1a_0}{F(S+G_1W+\Psi)} + \frac{G_1^2h_0a_0 - 2(\kappa+1)a_1^2G_1}{F\Psi(S+G_1W+\Psi)},
$$

$$
\begin{aligned}
A_5 = {} & \frac{\kappa+2-2t^2}{F}a_1 - \frac{(1-t^2)G_1(W+h_0)a_1}{F(S+G_1W+\Psi)} - (a_0' - a_1't)\frac{\sqrt{1-t^2}(S+a_0+a_1t)}{F(S+G_1W+\Psi)}\\
& - 2a_1\left(\frac{\kappa+2-t^2}{2F} - \frac{G_1(W+h_0)}{2F(S+G_1W+\Psi)} + \frac{(\kappa+2-2t^2)(S+a_0+a_1t)}{F(S+G_1W+\Psi)}\right)\\
& - \frac{2a_0t}{F} + \frac{G_1(W+h_0)t}{F(S+G_1W+\Psi)} - \frac{a_1[2(\kappa+1)a_1^2 - G_1h_0a_0]}{F\Psi(a_0+G_1h_0)},
\end{aligned}
$$

$$
B_1 = \frac{\kappa+1}{2F(R+G_1W+\Phi)},
$$

$$
\begin{aligned}
B_2 = {} & \frac{-2(\kappa+1)a_1}{F(R+G_1W+\Phi)} + \frac{(\kappa+2-3t^2)t}{2F} - \frac{t(1-2t^2)G_1(W+h_0)}{2F(R+G_1W+\Phi)}\\
& - \frac{t(\kappa+2-2t^2)(R+a_0-a_1t)}{F(R+G_1W+\Phi)} + \frac{t(G_1h_0a_0 - 2(\kappa+1)a_1^2)}{F\Phi(R+G_1W+\Phi)},
\end{aligned}
$$

$$
B_3 = \frac{4(\kappa+1)a_1 - tG_1(W+h_0)}{2F(R+G_1W+\Phi)} + \frac{(\kappa+2-t^2)t}{2F} + \frac{t(\kappa+2-2t^2)(R+a_0-a_1t)}{F(R+G_1W+\Phi)},
$$

$$
B_4 = -\frac{G_1a_0}{F(R+G_1W+\Phi)} + \frac{G_1^2h_0a_0 - 2(\kappa+1)a_1^2G_1}{F\Phi(R+G_1W+\Phi)},
$$

$$
\begin{aligned}
B_5 = {} & -\frac{\kappa+2-2t^2}{F}a_1 + \frac{(1-t^2)G_1(W+h_0)}{F(R+G_1W+\Phi)}a_1 + (a_0' + a_1't)\frac{\sqrt{1-t^2}(R+a_0-a_1t)}{F(R+G_1W+\Phi)}\\
& + 2a_1\left(\frac{\kappa+2-t^2}{2F} - \frac{G_1(W+h_0)}{2F(R+G_1W+\Phi)} + \frac{(\kappa+2-2t^2)(R+a_0-a_1t)}{F(R+G_1W+\Phi)}\right)\\
& - \frac{2a_0t}{F} + \frac{G_1(W+h_0)t}{F(R+G_1W+\Phi)} + \frac{a_1[2(\kappa+1)a_1^2 - G_1h_0a_0]}{F\Phi(a_0+G_1h_0)},
\end{aligned}
$$

and

$$A_6 = B_6 = \frac{\kappa+2}{F}a_0 + \frac{2(\kappa+1)a_1^2 - G_1 h_0 a_0}{F(a_0+G_1h_0)}. \tag{3.28}$$

The three eigenvalues of system (3.27) are

$$\lambda_0 = \frac{\sqrt{1-t^2}(R+S+2a_0)t^2}{F[2a_1t-(R-S)]}, \quad \lambda_+ = \frac{\sqrt{1-t^2}(S+a_0+a_1t)t^2}{F(S+G_1W+\Psi)},$$
$$\lambda_- = -\frac{\sqrt{1-t^2}(R+a_0-a_1t)t^2}{F(R+G_1W+\Phi)}, \tag{3.29}$$

and the three characteristics passing through point (ξ,η) are defined by

$$\begin{cases} \dfrac{\mathrm{d}z_i(t;\xi,\eta)}{\mathrm{d}t} = \lambda_i(t,z,W,R,S)(t,z_i(t;\xi,\eta)), & i=0,\pm. \\ z_i(\xi;\xi,\eta)=\eta, \end{cases} \tag{3.30}$$

In order to construct the integral equations later, we first show the solvability of the ODE problem (3.12), which is equivalent to the solvability of the following ODE problem:

$$\begin{cases} \dfrac{\mathrm{d}s(t)}{\mathrm{d}t} = \dfrac{s-b}{2t} + \widetilde{B}_1\dfrac{(s-b)^2}{t} + \widetilde{B}_2 b + \widetilde{B}_3 s + \widetilde{B}_4 th + \widetilde{B}_5 t^2 + \widetilde{B}_6 t, \\ s(0)=s'(0)=0, \end{cases} \tag{3.31}$$

where $\widetilde{B}_i$ $(i=1,\dots,6)$ are the functions B_i $(i=1,\dots,6)$ but with h, b, and $\tilde{z}(t)$ replacing W, R, and z, respectively. Actually, $s(t)$ is the boundary value of S on the curve $z=\tilde{z}(t)$. For the ODE problem (3.31), we have the following:

LEMMA 3.1. *Let* (3.24) *be satisfied. Then there exists a positive constant* $\delta_1 < t_0$ *such that the ODE problem* (3.31) *has a unique* C^2*-solution on* $t\in[0,\delta_1]$.

Proof. Throughout the paper, we use k_0 and K_0 to denote two positive constants depending only on the C^2 norms of $\hat{h}_0, \hat{a}_0, \hat{a}_1, \tilde{h}_0$, the C^3 norm of $\tilde{b}_0$, and the constants κ, ε_0, which may change from line to line.

Due to the last line in (3.24), we first have

$$|h(t)|\le K_0t^2,\ |h'(t)|\le K_0t,\ |b(t)|\le K_0t^2,\ |b'(t)|\le K_0t, \tag{3.32}$$

from which we can choose $\bar{\delta}_1$ small enough such that

$$\begin{aligned} &k_0\le F(t)\le K_0, \quad k_0\le G_1(t)\le K_0, \\ &b(t)+G_1h(t)+\Phi \ge a_0+G_1h_0-(|a_1|t+|b(t)|+G_1|h(t)|) \\ &\qquad \ge \varepsilon_0-(K_0\bar{\delta}_1+K_0\bar{\delta}_1^5+K_0\cdot K_0\bar{\delta}_1^3)\ge \tfrac{\varepsilon_0}{2} \quad \forall\ t\in[0,\bar{\delta}_1]. \end{aligned} \tag{3.33}$$

Let

$$M_1 = \max\{K_0, \max_{t\in[0,\bar{\delta}_1]}|\widetilde{B}_i|\ (i=1,\dots,6)\}. \tag{3.34}$$

We denote $s^{(0)}(t)=0$ and then define quantities $s^{(k)}(t)$ $(k\ge1)$ by the relation

$$\begin{aligned} s^{(k)}(t) = \int_0^t \Bigg\{ &\frac{s_2^{(k-1)}-b}{2\tau} + \widetilde{B}_1\frac{(s^{(k-1)}-b)^2}{\tau} + \widetilde{B}_2 b + \widetilde{B}_3 s_2^{(k-1)} \\ &+ \widetilde{B}_4\tau h + \widetilde{B}_5\tau^2 + \widetilde{B}_6\tau \Bigg\}\,\mathrm{d}\tau. \end{aligned} \tag{3.35}$$

By the standard argument of induction, we can prove that for all $k \geq 1$

$$\left|s^{(k)}(t)\right| \leq M_1 t^2 \sum_{i=0}^{k} \left(\frac{2}{3}\right)^j, \quad \left|s^{(k+1)}(t) - s^{(k)}(t)\right| \leq M_1 t^2 \left(\frac{2}{3}\right)^k \quad \forall\, t \in [0, \delta_1]$$

for the positive constant $\delta_1 = \min\{\bar{\delta}_1, 1/(4M_1)\}$, from which we know that the sequence $s^{(k)}(t)$ converges uniformly, and the limit function, denoted by $s(t)$, is continuous and satisfies $|s(t)| \leq 3M_1 t^2$ for any $t \in [0, \delta_1]$. The boundary conditions in (3.31) and the smoothness of $s(t)$ can be checked from (3.35). The lemma is proved. ∎

Next we define the domain and the admissible functions. Set

$$\overline{D}_{\delta_1} = \left\{(t, z)|\ t \in [0, \delta_1],\ \tilde{z}(t) \leq z \leq \frac{r_2 - r_1}{2}\right\},$$

which is a closed domain in the (t, z)-plane. Let $\mathcal{S}(\overline{D}_{\delta_1})$ be a function class incorporating all vector functions $\mathbf{F} = (f_1, f_2, f_3)^T : \overline{D}_{\delta_1} \to \mathbb{R}^3$ that satisfy the following properties:

$$\begin{aligned}
&(\mathrm{P}_1):\ f_i\ (i = 1, 2, 3) \text{ are continuous on } \overline{D}_{\delta_1};\\
&(\mathrm{P}_2): (f_1, f_2, f_3)^T(0, z) = (0, 0, 0)\ \forall\, z \in [0, \tfrac{r_2 - r_1}{2}];\\
&(\mathrm{P}_3): (f_1, f_2, f_3)^T(t, \tilde{z}(t)) = (h, b, s)^T(t)\ \forall\, t \in [0, \delta_1];\\
&(\mathrm{P}_4): \max_{(t,z)\in\overline{D}_{\delta_1}} |f_i(t, z)| \leq \widetilde{M} t^2\ (i = 1, 2, 3),
\end{aligned} \tag{3.36}$$

where $\widetilde{M}$ $(\geq 3M_1)$ is a fixed constant.

We note by Lemma 3.1 that the vector function $(h, b, s)^T$ belongs to $\mathcal{S}(\overline{D}_{\delta_1})$, which means that $\mathcal{S}(\overline{D}_{\delta_1})$ is not empty. For any $(f_1, f_2, f_3)^T \in \mathcal{S}(\overline{D}_{\delta_1})$, by (3.24) and the property (P_4) in (3.36), we can choose $\delta_2 < \min\{\delta_1, 1/\widetilde{M}\}$ small enough such that for all $(t, z) \in \overline{D}_{\delta_1} \cap \{t \leq \delta_2\}$

$$2a_1 t - (f_2 - f_3) \geq \varepsilon_0 t, \quad f_3 + G_1 f_1 + \Psi \geq \frac{\varepsilon_0}{2}, \quad f_2 + G_1 f_1 + \Phi \geq \frac{\varepsilon_0}{2}, \tag{3.37}$$

and

$$\begin{gathered}
\underline{k} \leq \frac{\sqrt{1 - t^2}(f_2 + f_3 + 2a_0)t}{F[2a_1 t - (f_2 - f_3)]} \leq \overline{K}, \quad \underline{k} \leq \frac{\sqrt{1 - t^2}(f_3 + a_0 + a_1 t)}{F(f_3 + G_1 f_1 + \Psi)} \leq \overline{K},\\
\underline{k} \leq \frac{\sqrt{1 - t^2}(f_2 + a_0 - a_1 t)}{F(f_2 + G_1 f_1 + \Phi)} \leq \overline{K},
\end{gathered} \tag{3.38}$$

and

$$\underline{k} \geq 2\overline{K}\delta_2, \quad \frac{r_2 - r_1}{2} \geq \tilde{z}(\delta_2) + \overline{K}\delta_2^3 := z_1 \tag{3.39}$$

for some positive constants $\underline{k}$ and $\overline{K}$. We claim that the number z_1 is positive. Indeed, one recalls (3.9) and the expression $\tilde{z}(t) = r_2 - \tilde{r}(t)$ and then uses the transformation (3.22) to obtain

$$\tilde{z}(\delta_2) = -\int_0^{\delta_2} \frac{\sqrt{1 - t^2}\tilde{b}_0 t^2}{F(\tilde{b}_0 + G_1 \tilde{h}_0)}\, \mathrm{d}t = -\int_0^{\delta_2} \frac{\sqrt{1 - t^2}[b + a_0 - a_1 t]t^2}{F[b + G_1 h + a_0 - a_1 t + G_1 h_0]}\, \mathrm{d}t,$$

which combined with the property (P_3) in (3.36) and (3.38) yields

$$\tilde{z}(\delta_2) \geq -\int_0^{\delta_2} \overline{K} t^2 \, \mathrm{d}t = -\frac{\overline{K}}{3}\delta_2^3,$$

from which we observe that $z_1 \geq 2\overline{K}\delta_2^3/3 > 0$. Denote $\bar{z}(t) = z_1 - \overline{K}t^3$. Then for any $t \in [0, \delta_2]$, one has $\bar{z}(t) \geq \tilde{z}(t)$ and $\bar{z}(t) = \tilde{z}(t)$ iff $t = \delta_2$. Moreover, it is easy to see by (3.29) and (3.38) that, for any $\mathbf{F} \in \mathcal{S}(\overline{D}_{\delta_1})$,

$$\lambda_0(\mathbf{F}) - \lambda_-(\mathbf{F}) \geq \underline{k}t, \quad \lambda_+(\mathbf{F}) - \lambda_-(\mathbf{F}) \geq \underline{k}t^2. \tag{3.40}$$

Let $\overline{D}_{\delta_2}$ be a closed domain defined by

$$\overline{D}_{\delta_2} = \{(t, z) | \; t \in [0, \delta_2], \; \tilde{z}(t) \leq z \leq \bar{z}(t)\} \tag{3.41}$$

and $\mathcal{S}(\overline{D}_{\delta_2})$ be the corresponding function class of $\mathcal{S}(\overline{D}_1)$, but with $\overline{D}_{\delta_2}$ replacing $\overline{D}_{\delta_1}$. We point out that the domain $\overline{D}_{\delta_2}$ is a strong determinate domain for system (3.27), that is, for any vector function $\mathbf{F} = (f_1, f_2, f_3)^T \in \mathcal{S}(\overline{D}_{\delta_2})$ and for any point $(\xi, \eta) \in \overline{D}_{\delta_2}$, the characteristic curves $z_i(t; \xi, \eta)$ $(i = 0, \pm)$ stay inside $\overline{D}_{\delta_2}$ until the intersection with the boundary curves $z = \tilde{z}(t)$ or $t = 0$. Here $z_i(t; \xi, \eta)$ $(i = 0, \pm)$ are defined in (3.30) but with $(f_1, f_2, f_3)^T$ replacing $(W, R, S)^T$ in λ_i $(i = 0, \pm)$. We use ξ_i $(i = 0, +)$ to denote the intersection times of curves $z = z_i(t; \xi, \eta)$ $(i = 0, +)$ and the boundary of $\overline{D}_{\delta_2}$. We claim that

$$z_0(t; \xi, \eta) < z_+(t; \xi, \eta) \tag{3.42}$$

holds for $t \in [\xi_0, \xi)$. In fact, from (3.30), one has

$$z_+(t; \xi, \eta) = \eta - \int_t^{\xi} \frac{\sqrt{1 - t^2}(f_3 + a_0 + a_1 t)}{F(f_3 + G_1 f_1 + \Psi)} t^2 \, \mathrm{d}t,$$
$$z_0(t; \xi, \eta) = \eta - \int_t^{\xi} \frac{\sqrt{1 - t^2}(f_2 + f_3 + 2a_0)t}{F[2a_1 t - (f_2 - f_3)]} t \, \mathrm{d}t,$$

which together with (3.38) and (3.39) gains

$$\begin{aligned} &z_+(t; \xi, \eta) - z_0(t; \xi, \eta) \\ &\quad = \int_t^{\xi} \frac{\sqrt{1 - t^2}(f_2 + f_3 + 2a_0)t}{F[2a_1 t - (f_2 - f_3)]} t \, \mathrm{d}t - \int_t^{\xi} \frac{\sqrt{1 - t^2}(f_3 + a_0 + a_1 t)}{F(f_3 + G_1 f_1 + \Psi)} t^2 \, \mathrm{d}t \\ &\quad \geq \int_t^{\xi} \underline{k}t \, \mathrm{d}t - \int_t^{\xi} \overline{K}t^2 \, \mathrm{d}t \geq (\xi^2 - t^2)\left[\frac{\underline{k}}{2} - \frac{\overline{K}}{3}(\xi + t)\right] \\ &\quad \geq \frac{\xi^2 - t^2}{2}\left[\underline{k} - \frac{4\overline{K}}{3}\delta_2\right] \geq \frac{\xi^2 - t^2}{3}\overline{K}\delta_2 > 0, \end{aligned}$$

which leads to (3.42). Furthermore, it follows by employing (3.38) again that for $t \in [\xi_+, \xi]$

$$\left|z_+(t; \xi, \eta) - z_-(t; \xi, \eta)\right| \leq \int_0^{\xi} 2\overline{K}t^2 \, \mathrm{d}t = \frac{2}{3}\overline{K}\xi^3. \tag{3.43}$$

Combining with (3.43) and (3.28) yields

$$\left|A_6(t, z_+(t; \xi, \eta)) - B_6(t, z_-(t; \xi, \eta))\right| \leq K_0\left|z_+(t; \xi, \eta) - z_-(t; \xi, \eta)\right| \leq K_0\xi^3. \tag{3.44}$$

3.2.2. The integral equations and iterative sequence. We proceed to construct the integral equations, based on the differential equations (3.27). For any $(\xi,\eta)\in\overline{D}_{\delta_2}$, we integrate the system (3.27) along the characteristic curves $z=z_i(t)$ defined in (3.30) and use (3.23) to obtain a system of integral equations

$$\begin{cases} W(\xi,\eta)=h(\xi_0)+\displaystyle\int_{\xi_0}^{\xi} C_1 t(t,z_0(t))\,\mathrm{d}t, \\ R(\xi,\eta)=b(\xi_+)+\displaystyle\int_{\xi_+}^{\xi}\Big\{\frac{R-S}{2t}+A_1\frac{(R-S)^2}{t}+A_2R \\ \qquad\qquad +A_3S+A_4tW+A_5t^2+A_6t\Big\}(t,z_+(t))\,\mathrm{d}t, \\ S(\xi,\eta)=\displaystyle\int_0^{\xi}\Big\{\frac{S-R}{2t}+B_1\frac{(R-S)^2}{t}+B_2R \\ \qquad\qquad +B_3S+B_4tW+B_5t^2+B_6t\Big\}(t,z_-(t))\,\mathrm{d}t, \end{cases} \tag{3.45}$$

where ξ_i $(i=0,+)$ are the intersection times of curves $z=z_i(t;\xi,\eta)$ $(i=0,+)$ and the boundary of $\overline{D}_{\delta_2}$. Here we used the fact that the negative characteristic curve $z=z_-(t)=z_-(t;\xi,\eta)$ only intersects the line $t=0$. We note that λ_i $(i=0,\pm)$ depend on the solution (W,R,S); then the characteristic curves $z=z_i(t)$ $(i=0,\pm)$ and also the numbers ξ_0,ξ_+ may change in each iteration step, which makes the construction of iterative sequence more complicated and the proof of the convergence of the iterative sequence rather more difficult.

We now construct an iterative sequence for the integral system (3.45). Set $W^{(0)}(t,z)=h(t)$, $R^{(0)}(t,z)=b(t)$, and $S^{(0)}(t,z)=s(t)$. Let $z=\hat{z}_i^{(0)}(t)$ $(i=0,+)$ be the curves in $\overline{D}_{\delta_2}$ defined by

$$\begin{cases} \dfrac{\mathrm{d}\hat{z}_i^{(0)}(t)}{\mathrm{d}t}=\lambda_i(t,z,W^{(0)},R^{(0)},S^{(0)})(t,\hat{z}_i^{(0)}(t)), & i=0,+, \\ \hat{z}_i^{(0)}(0)=0, \end{cases}$$

where λ_i $(i=0,+)$ are given in (3.29). It is easily seen by (3.39) that $\hat{z}_+^{(0)}(t)<\hat{z}_0^{(0)}(t)$ for $t>0$. Then $\overline{D}_{\delta_2}$ is divided into three disjoint subdomains,

$$\overline{D}_{\delta_2}=D_{\delta_2}^{(01)}\cup D_{\delta_2}^{(02)}\cup D_{\delta_2}^{(03)},$$

where $D_{\delta_2}^{(01)}=\{(t,z)|\ z\geq\hat{z}_0^{(0)}(t)\}\cap\overline{D}_{\delta_2}$, $D_{\delta_2}^{(02)}=\{(t,z)|\ \hat{z}_+^{(0)}(t)\leq z<\hat{z}_0^{(0)}(t)\}\cap\overline{D}_{\delta_2}$, and $D_{\delta_2}^{(03)}=\{(t,z)|\ z<\hat{z}_+^{(0)}(t)\}\cap\overline{D}_{\delta_2}$. For any $(\xi,\eta)\in\overline{D}_{\delta_2}$, we define the characteristic curves $z=z_i^{(0)}(t)=:z_i^{(0)}(t;\xi,\eta)$ $(i=0,\pm)$ as

$$\begin{cases} \dfrac{\mathrm{d}z_i^{(0)}(t;\xi,\eta)}{\mathrm{d}t}=\lambda_i(t,z,W^{(0)},R^{(0)},S^{(0)})(t,z_i^{(0)}(t;\xi,\eta)), & i=0,\pm. \\ z_i^{(0)}(\xi;\xi,\eta)=\eta, \end{cases}$$

Thus, if $(\xi,\eta)\in D_{\delta_2}^{(01)}$, then $z=z_i^{(0)}(t;\xi,\eta)$ $(i=0,\pm)$ intersect $t=0$; if $(\xi,\eta)\in D_{\delta_2}^{(02)}$, then $z=z_i^{(0)}(t;\xi,\eta)$ $(i=\pm)$ intersect $t=0$ while $z=z_0^{(0)}(t;\xi,\eta)$ intersects $z=\tilde{z}(t)$; if $(\xi,\eta)\in D_{\delta_2}^{(03)}$, then $z=z_-^{(0)}(t;\xi,\eta)$ intersects $t=0$ while $z=z_i^{(0)}(t;\xi,\eta)$ $(i=0,+)$

intersect $z = \tilde{z}(t)$. According to the above three cases, for any $(\xi, \eta) \in \overline{D}_{\delta_2}$, we construct the functions $(W^{(1)}, R^{(1)}, S^{(1)})(\xi, \eta)$ as follows:

(3.46)
$$\begin{cases} W^{(1)}(\xi,\eta) = h(\xi_0^{(0)}) + \displaystyle\int_{\xi_0^{(0)}}^{\xi} C_1^{(0)} t(t, z_0^{(0)}(t))\ \mathrm{d}t, \\ R^{(1)}(\xi,\eta) = b(\xi_+^{(0)}) + \displaystyle\int_{\xi_+^{(0)}}^{\xi} \Big\{ \frac{R^{(0)} - S^{(0)}}{2t} + A_1^{(0)} \frac{(R^{(0)} - S^{(0)})^2}{t} \\ \qquad + A_2^{(0)} R^{(0)} + A_3^{(0)} S^{(0)} + A_4^{(0)} t W^{(0)} + A_5^{(0)} t^2 + A_6^{(0)} t \Big\} (t, z_+^{(0)}(t))\ \mathrm{d}t, \\ S^{(1)}(\xi,\eta) = \displaystyle\int_0^{\xi} \Big\{ \frac{S^{(0)} - R^{(0)}}{2t} + B_1^{(0)} \frac{(R^{(0)} - S^{(0)})^2}{t} + B_2^{(0)} R^{(0)} \\ \qquad + B_3^{(0)} S^{(0)} + B_4^{(0)} t W^{(0)} + B_5^{(0)} t^2 + B_6^{(0)} t \Big\} (t, z_-^{(0)}(t))\ \mathrm{d}t, \end{cases}$$

where $\xi_i^{(0)}$ $(i = 0, +)$ are the intersection times of curves $z = z_i^{(0)}(t; \xi, \eta)$ $(i = 0, +)$ and the boundary of $\overline{D}_{\delta_2}$ satisfying $\xi_0^{(0)} = 0$ if $\eta \geq \hat{z}_0^{(0)}(t)$, $\xi_0^{(0)} > 0$ if $\eta < \hat{z}_0^{(0)}(t)$ and $\xi_+^{(0)} = 0$ if $\eta \geq \hat{z}_+^{(0)}(t)$, $\xi_+^{(0)} > 0$ if $\eta < \hat{z}_+^{(0)}(t)$. The functions $C_1^{(0)}$, $A_i^{(0)}$, and $B_i^{(0)}$ $(i = 1, \ldots, 6)$ in (3.46) are functions C_1, A_i, and B_i $(i = 1, \ldots, 6)$ given in (3.45) but with $W^{(0)}$, $R^{(0)}$, and $S^{(0)}$ replacing W, R, and S, respectively. We note by (3.42) that $z_0^{(0)}(t) < z_+^{(0)}(t)$ for $t \in [\xi_0^{(0)}, \xi)$.

After defining the functions $(W^{(k)}, R^{(k)}, S^{(k)})(t, z)$ $(k \geq 1)$, we can define the characteristic curves $z = \hat{z}_i^{(k)}(t)$ $(i = 0, +)$ passing through $(0, 0)$ and $z = z_i^{(k)}(t) =: z_i^{(k)}(t; \xi, \eta)$ $(i = 0, \pm)$ passing through $(\xi, \eta) \in \overline{D}_{\delta_2}$ as

$$\begin{cases} \dfrac{\mathrm{d}\hat{z}_i^{(k)}(t)}{\mathrm{d}t} = \lambda_i(t, z, W^{(k)}, R^{(k)}, S^{(k)})(t, \hat{z}_i^{(k)}(t)), \\ \hat{z}_i^{(k)}(0) = 0, \end{cases} \qquad i = 0, +,$$

and

$$\begin{cases} \dfrac{\mathrm{d}z_i^{(k)}(t;\xi,\eta)}{\mathrm{d}t} = \lambda_i(t, z, W^{(k)}, R^{(k)}, S^{(k)})(t, z_i^{(k)}(t;\xi,\eta)), \\ z_i^{(k)}(\xi;\xi,\eta) = \eta, \end{cases} \qquad i = 0, \pm. \tag{3.47}$$

The domain $\overline{D}_{\delta_2}$ now is divided into three disjoint subdomains,

$$\overline{D}_{\delta_2} = D_{\delta_2}^{(k1)} \cup D_{\delta_2}^{(k2)} \cup D_{\delta_2}^{(k3)},$$

where $D_{\delta_2}^{(k1)} = \{(t,z) |\ z \geq \hat{z}_0^{(k)}(t)\} \cap \overline{D}_{\delta_2}$, $D_{\delta_2}^{(k2)} = \{(t,z) |\ \hat{z}_+^{(k)}(t) \leq z < \hat{z}_0^{(k)}(t)\} \cap \overline{D}_{\delta_2}$, and $D_{\delta_2}^{(k3)} = \{(t,z) |\ z < \hat{z}_+^{(k)}(t)\} \cap \overline{D}_{\delta_2}$. As in the previous analysis, for any $(\xi, \eta) \in \overline{D}_{\delta_2}$, the curve $z = z_-^{(k)}(t; \xi, \eta)$ always intersects $t = 0$. Moreover, if $(\xi, \eta) \in D_{\delta_2}^{(k1)}$, then both curves $z = z_i^{(k)}(t; \xi, \eta) (i = 0, +)$ intersect $t = 0$; if $(\xi, \eta) \in D_{\delta_2}^{(k2)}$, then the curve $z = z_+^{(k)}(t; \xi, \eta)$ intersects $t = 0$ and the curve $z = z_0^{(k)}(t; \xi, \eta)$ intersects $z = \tilde{z}(t)$; if $(\xi, \eta) \in D_{\delta_2}^{(k3)}$, then both curves $z = z_i^{(k)}(t; \xi, \eta) (i = 0, +)$ intersect $z = \tilde{z}(t)$. We use $\xi_i^{(k)}$ $(i = 0, +)$ to represent the intersection times of curves $z = z_i^{(k)}(t; \xi, \eta)$ $(i = 0, +)$

and the boundary of $\overline{D}_{\delta_2}$. Then the numbers $\xi_i^{(k)}$ $(i=0,+)$ satisfy $\xi_0^{(k)}=0$ if $\eta\geq\hat{z}_0^{(k)}(t)$, $\xi_0^{(k)}>0$ if $\eta<\hat{z}_0^{(k)}(t)$ and $\xi_+^{(k)}=0$ if $\eta\geq\hat{z}_+^{(k)}(t)$, $\xi_+^{(k)}>0$ if $\eta<\hat{z}_+^{(k)}(t)$. Thus we define the quantities $(W^{(k+1)},R^{(k+1)},S^{(k+1)})(t,z)$ by the relations

$$\begin{cases} W^{(k+1)}(\xi,\eta)=h(\xi_0^{(k)})+\displaystyle\int_{\xi_0^{(k)}}^{\xi} C_1^{(k)}t(t,z_0^{(k)}(t))\,\mathrm{d}t,\\ R^{(k+1)}(\xi,\eta)=b(\xi_+^{(k)})+\displaystyle\int_{\xi_+^{(k)}}^{\xi}\left\{\frac{R^{(k)}-S^{(k)}}{2t}+A_1^{(k)}\frac{(R^{(k)}-S^{(k)})^2}{t}\right.\\ \qquad\left.+A_2^{(k)}R^{(k)}+A_3^{(k)}S^{(k)}+A_4^{(k)}tW^{(k)}+A_5^{(k)}t^2+A_6^{(k)}t\right\}(t,z_+^{(k)}(t))\,\mathrm{d}t,\\ S^{(k+1)}(\xi,\eta)=\displaystyle\int_0^{\xi}\left\{\frac{S^{(k)}-R^{(k)}}{2t}+B_1^{(k)}\frac{(R^{(k)}-S^{(k)})^2}{t}+B_2^{(k)}R^{(k)}\right.\\ \qquad\left.+B_3^{(k)}S^{(k)}+B_4^{(k)}tW^{(k)}+B_5^{(k)}t^2+B_6^{(k)}t\right\}(t,z_-^{(k)}(t))\,\mathrm{d}t, \end{cases} \tag{3.48}$$

where $C_1^{(k)}$, $A_i^{(k)}$, and $B_i^{(k)}$ $(i=1,\ldots,6)$ are functions C_1, A_i, and B_i $(i=1,\ldots,6)$ given in (3.45) but with $W^{(k)}$, $R^{(k)}$, and $S^{(k)}$ replacing W, R, and S, respectively.

We shall show that the sequences $(W^{(k)},R^{(k)},S^{(k)})$ converge uniformly in a domain $\overline{D}_\delta\subset\overline{D}_{\delta_2}$ for some small $\delta<\delta_2$.

3.2.3. Several key lemmas. In view of the expressions of C_1, A_i, and B_i $(i=1,\ldots,6)$ given in (3.27), if $(W,R,S)^T\in\mathcal{S}(\overline{D}_{\delta_2})$, then by (3.36) and (3.37) there exists a positive constant $\overline{M}$ such that for all $(t,z)\in\overline{D}_{\delta_2}$

$$\begin{aligned} &|C_1(t,z,R,S)|\leq\overline{M}, && |A_i(t,z,S,W)|\leq\overline{M}, && |B_i(t,z,R,W)|\leq\overline{M},\\ &|C_{1z}|;t|C_{1R}|;t|C_{1S}|\leq\overline{M}, && |A_{iz}|;|A_{iS}|;|A_{iW}|\leq\overline{M}, && |B_{iz}|;|B_{iR}|;|B_{iW}|\leq\overline{M}. \end{aligned} \tag{3.49}$$

We now choose M and δ satisfying

$$M=\max\{1,4M_1,4K_0,4\widehat{K},\overline{M}\},\quad \delta\leq\left\{\delta_2,\frac{1}{100M},\frac{\underline{k}}{2M}\right\}, \tag{3.50}$$

such that there hold

$$\left(\frac{1}{2}+13M\delta\right)\exp(2M\delta^2)\leq\frac{2}{3},\ 4M\delta+\frac{M\delta}{\underline{k}}<\frac{2}{3}. \tag{3.51}$$

Here the constants M_1, $\widehat{K}$, and $\underline{k},\overline{K}$ are, respectively, are given in (3.34), (3.26), and (3.38). We denote

$$\overline{D}_\delta=\{(t,z)|\ 0\leq t\leq\delta,\ \tilde{z}(t)\leq z\leq\tilde{z}(\delta)+\overline{K}\delta^3-\overline{K}t^3\}.$$

It is clear that $\overline{D}_\delta\subset\overline{D}_{\delta_2}$. Then we have the following.

LEMMA 3.2. *For all $k\geq1$ the inequalities*

$$\begin{aligned} &\left|W^{(k)}(\xi,\eta)\right|;\ \left|R^{(k)}(\xi,\eta)\right|;\ \left|S^{(k)}(\xi,\eta)\right|\leq M\xi^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j,\\ &\left|R^{(k)}(\xi,\eta)-S^{(k)}(\xi,\eta)\right|\leq M\xi^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j \end{aligned} \tag{3.52}$$

hold in $\overline{D}_\delta$.

Proof. We use the standard argument of induction to prove the lemma. First we show that all inequalities in (3.52) hold for $n = 1$. Second, we assume that they are all true for $n = k$ and then establish each inequality for $n = k + 1$.

Since $(W^{(0)}, R^{(0)}, S^{(0)})^T = (h, b, s)^T \in \mathcal{S}(\overline{D}_{\delta_2})$, then by (3.49) and (3.50) one has

$$|C_1^{(0)}(t,z)|; |A_i^{(0)}(t,z)|; |B_i^{(0)}(t,z)| \leq M \ (i = 1, \dots, 6) \quad \forall\ (t,z) \in \overline{D}_{\delta_2}. \tag{3.53}$$

From (3.32) and (3.53), we derive by (3.46)

$$\begin{aligned} |W^{(1)}(\xi,\eta)| &\leq |h(\xi_0^{(0)})| + \int_{\xi_0^{(0)}}^{\xi} t \cdot |C_1^{(0)}(t, z_0^{(0)}(t))| \, \mathrm{d}t \\ &\leq K_0(\xi_0^{(0)})^2 + \int_0^{\xi} Mt \, \mathrm{d}t \leq \frac{M}{4}\xi^2 + \frac{M}{2}\xi^2 \leq M\xi^2 \sum_{j=0}^{1} \left(\frac{2}{3}\right)^j. \end{aligned} \tag{3.54}$$

For the quantity $R^{(1)}(\xi,\eta)$, one has by noting $M \geq 4K_0$

$$\begin{aligned} &|R^{(1)}(\xi,\eta)| \\ &\leq |b(\xi_+^{(0)})| + \int_{\xi_+^{(0)}}^{\xi} \left\{ \frac{|R^{(0)} - S^{(0)}|}{2t} + |A_1^{(0)}| \frac{(R^{(0)} - S^{(0)})^2}{t} + |A_2^{(0)}| \cdot |R^{(0)}| \right. \\ &\qquad \left. + |A_3^{(0)}| \cdot |S^{(0)}| + |A_4^{(0)}| \cdot t \cdot |W^{(0)}| + |A_5^{(0)}| t^2 + |A_6^{(0)}| t \right\} (t, z_+^{(0)}(t)) \, \mathrm{d}t \\ &\leq \frac{M}{4}\xi^2 + \int_0^{\xi} \left\{ \frac{M}{2}t + M \cdot M^2 t^3 + 3M \cdot Mt^2 + Mt^2 + Mt \right\} \mathrm{d}t \\ &\leq M\xi^2 \left\{ 1 + \left(\delta + M\delta + M^2\delta^2 \right) \right\} \leq M\xi^2 \sum_{j=0}^{1} \left(\frac{2}{3}\right)^j. \end{aligned} \tag{3.55}$$

The above estimate also holds for $S^{(1)}(\xi,\eta)$. We next estimate the term $|R^{(1)}(\xi,\eta) - S^{(1)}(\xi,\eta)|$ by (3.53),

$$|R^{(1)}(\xi,\eta) - S^{(1)}(\xi,\eta)| \leq |b(\xi_+^{(0)})| + I_1 + I_2, \tag{3.56}$$

where

$$\begin{aligned} I_1 = \int_{\xi_+^{(0)}}^{\xi} &\left\{ 2\frac{|R^{(0)} - S^{(0)}|}{2t} + 2M \frac{(R^{(0)} - S^{(0)})^2}{t} + 2M \cdot (|R^{(0)}| + |S^{(0)}|) \right. \\ &\left. + 2M \cdot |W^{(0)}| + 2Mt^2 + |A_6^{(0)}(t, z_+^{(0)}(t)) - B_6^{(0)}(t, z_-^{(0)}(t))| \cdot t \right\} \mathrm{d}t, \\ I_2 = \int_0^{\xi_+^{(0)}} &\left\{ \frac{|R^{(0)} - S^{(0)}|}{2t} + |B_1^{(0)}| \cdot \frac{(R^{(0)} - S^{(0)})^2}{t} + |B_2^{(0)}| \cdot |R^{(0)}| + |B_3^{(0)}| \cdot |S^{(0)}| \right. \\ &\left. + |B_4^{(0)}| \cdot t \cdot |W^{(0)}| + |B_5^{(0)}| \cdot t^2 + |B_6^{(0)}| \cdot t \right\} (t, z_-^{(0)}(t)) \, \mathrm{d}t. \end{aligned}$$

For I_1, it is observed by (3.45) that

$$\begin{aligned} I_1 &\leq \int_{\xi_+^{(0)}}^{\xi} \left\{ Mt + 2M \cdot M^2 t^3 + 6M \cdot Mt^2 + 2Mt^2 + M\xi^3 \cdot t \right\} \mathrm{d}t \\ &\leq M\xi^2 \left(\frac{1}{2} + \delta + 2M\delta + M^2\delta^2 + M\delta^3 \right) - \frac{1}{2}M(\xi_+^{(0)})^2. \end{aligned} \tag{3.57}$$

Moreover, we use (3.32) and (3.53) again to estimate I_2,

$$\begin{aligned} I_2 \le & \int_0^{\xi_+^{(0)}} \left\{\frac{1}{2}Mt + M\cdot M^2t^3 + 3M\cdot Mt^2 + M\cdot t^2 + M\cdot t\right\} \mathrm{d}t \\ \le & \frac{3}{4}M(\xi_+^{(0)})^2 + M\xi^2(M^2\delta^2 + M\delta + \delta). \end{aligned} \tag{3.58}$$

Inserting (3.57) and (3.58) into (3.56) and applying (3.32) gets

$$\begin{aligned} \left|R^{(1)}(\xi,\eta) - S^{(1)}(\xi,\eta)\right| \le & \frac{1}{4}M\xi^2 + M\xi^2\left(\frac{1}{2} + \delta + 2M\delta + M^2\delta^2 + M\delta^3\right) \\ & - \frac{1}{2}M(\xi_+^{(0)})^2 + \frac{3}{4}M(\xi_+^{(0)})^2 + M\xi^2(M^2\delta^2 + M\delta + \delta) \\ \le & \ M\xi^2\left\{1 + \left(2\delta + 3M\delta + 2M^2\delta^2 + M\delta^3\right)\right\} \le M\xi^2\sum_{j=0}^{1}\left(\frac{2}{3}\right)^j. \end{aligned} \tag{3.59}$$

We combine (3.54), (3.55), and (3.59) to obtain (3.52) for $k=1$.

Assume that (3.52) are valid for $n=k$. Then one has by the choice of $\widetilde{M} \ge 3M$

$$\left|W^{(k)}(\xi,\eta)\right| \le M\xi^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j \le 3M\xi^2 \le \widetilde{M}\xi^2,$$

and similarly

$$\left|R^{(k)}(\xi,\eta)\right|; \left|S^{(k)}(\xi,\eta)\right| \le \widetilde{M}\xi^2,$$

which along with the constructions of $(W^{(k)}, R^{(k)}, S^{(k)})$ in (3.48) indicates that

$$(W^{(k)}, R^{(k)}, S^{(k)})^T \in \mathcal{S}(\overline{D}_{\delta_2}).$$

Therefore, we recall (3.49) to find that

$$\left|C_1^{(k)}(t,z)\right|; \left|A_i^{(k)}(t,z)\right|; \left|B_i^{(k)}(t,z)\right| \le M \ (i=1,\ldots,6) \quad \forall\ (t,z)\in\overline{D}_{\delta_2}. \tag{3.60}$$

Thus for $n=k+1$, one has by (3.48) and (3.60)

$$\begin{aligned} \left|W^{(k+1)}(\xi,\eta)\right| \le & \left|h(\xi_0^{(k)})\right| + \int_{\xi_0^{(k)}}^{\xi} t\cdot\left|C_1^{(k)}(t, z_0^{(k)}(t))\right| \mathrm{d}t \\ \le & \frac{M}{4}\xi^2 + \int_0^{\xi} Mt\ \mathrm{d}t \le \frac{3}{4}M\xi^2 \le M\xi^2\sum_{j=0}^{k+1}\left(\frac{2}{3}\right)^j. \end{aligned} \tag{3.61}$$

We next derive the estimate of $R^{(k+1)}(\xi,\eta)$. Employing (3.32) and (3.60) again, it follows from the induction assumptions that

$$
\begin{aligned}
&\left|R^{(k+1)}(\xi,\eta)\right| \\
&\le \left|b(\xi_+^{(k)})\right| + \int_0^\xi \left\{ \frac{\left|R^{(k)}-S^{(k)}\right|}{2t} + \left|A_1^{(k)}\right| \frac{(R^{(k)}-S^{(k)})^2}{t} + \left|A_2^{(k)}\right| \cdot \left|R^{(k)}\right| \right. \\
&\quad \left. + \left|A_3^{(k)}\right| \cdot \left|S^{(k)}\right| + \left|A_4^{(k)}\right| \cdot t \cdot \left|W^{(k)}\right| + \left|A_5^{(k)}\right| t^2 + \left|A_6^{(k)}\right| t \right\} (t, z_+^{(k)}(t))\,\mathrm{d}t \\
&\le \frac{M}{4}\xi^2 + \int_0^\xi \left\{ \frac{M}{2} \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j t + M^3 \left(\sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \right)^2 t^3 \right. \\
&\qquad \left. + 3M^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j t^2 + Mt^2 + Mt \right\} \mathrm{d}t
\end{aligned}
$$

$$
\le M\xi^2 \left\{ \left(\frac{1}{4} + \frac{1}{2} + \delta\right) + \left(\frac{1}{4} + M^2\delta^2 + M\delta\right) \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \right\} \le M\xi^2 \sum_{j=0}^{k+1} \left(\frac{2}{3}\right)^j. \tag{3.62}
$$

The derivation in (3.62) is also valid for $S^{(k+1)}(\xi,\eta)$. For the term $|R^{(k+1)}(\xi,\eta) - S^{(k+1)}(\xi,\eta)|$, we compute by (3.48)

$$
\left|R^{(k+1)}(\xi,\eta) - S^{(k+1)}(\xi,\eta)\right| \le \left|b_1(\xi_+^{(k)})\right| + I_3 + I_4, \tag{3.63}
$$

where

$$
\begin{aligned}
I_3 = &\int_{\xi_+^{(k)}}^\xi \left\{ 2\frac{\left|R^{(k)}-S^{(k)}\right|}{2t} + 2M\frac{(R^{(k)}-S^{(k)})^2}{t} + 2M \cdot \left|R^{(k)}\right| + 2M \cdot \left|S^{(k)}\right| \right. \\
&\left. + 2M \cdot \left|W^{(k)}\right| + 2Mt^2 + \left|A_6^{(k)}(t, z_+^{(k)}(t)) - B_6^{(k)}(t, z_-^{(k)}(t))\right| \cdot t \right\} \mathrm{d}t, \\
I_4 = &\int_0^{\xi_+^{(k)}} \left\{ \frac{\left|S^{(k)}-R^{(k)}\right|}{2t} + \left|B_1^{(k)}\right| \frac{(R^{(k)}-S^{(k)})^2}{t} + \left|B_2^{(k)}\right| \cdot \left|R^{(k)}\right| \right. \\
&\left. + \left|B_3^{(k)}\right| \cdot \left|S^{(k)}\right| + \left|B_4^{(k)}\right| \cdot t \cdot \left|W^{(k)}\right| + \left|B_5^{(k)}\right| t^2 + \left|B_6^{(k)}\right| t \right\} (t, z_-^{(k)}(t))\,\mathrm{d}t.
\end{aligned}
$$

Using similar arguments as for I_1 and I_2, one deduces by the induction assumptions

$$
\begin{aligned}
I_3 \le &\int_{\xi_+^{(k)}}^\xi \left\{ Mt \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j + 2M^3 \left(\sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \right)^2 t^3 \right. \\
&\qquad \left. + 6M^2t^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j + 2Mt^2 + M\xi^3 \cdot t \right\} \mathrm{d}t \\
&\le M\xi^2 \left\{ (\delta + \delta^3) + \left(\frac{1}{2} + 2M^2\delta^2 + 2M\delta\right) \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \right\} \\
&\quad - \frac{1}{2} M (\xi_+^{(k)})^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j
\end{aligned} \tag{3.64}
$$

and

$$
\begin{aligned}
I_4 \le & \int_0^{\xi_+^{(k)}} \left\{ \frac{1}{2}Mt\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + M^3t^3\left(\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right)^2 \right.\\
& \left. \qquad + 3M^2t^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + Mt^2 + Mt \right\} \mathrm{d}t \\
\le & \left\{ M\xi^3 + \frac{1}{2}M\xi^2 + \left(\frac{1}{4}M(\xi_+^{(k)})^2 + M^3\xi^4 + M^2\xi^3\right)\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j \right\}. \qquad (3.65)
\end{aligned}
$$

We put (3.64) and (3.65) into (3.63) to acquire

$$
\begin{aligned}
& \left|R^{(k+1)}(\xi,\eta) - S^{(k+1)}(\xi,\eta)\right| \\
& \le M\xi^2\left\{ \left(\frac{3}{4} + 2\delta + \delta^3\right) + \left(\frac{1}{2} + 3M^2\delta^2 + 3M\delta\right)\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j \right\} \\
& \le M\xi^2\sum_{j=0}^{k+1}\left(\frac{2}{3}\right)^j. \qquad (3.66)
\end{aligned}
$$

Summing up (3.61)–(3.66) completes the proof of the induction step. Hence the lemma is proved. ∎

From Lemma 3.2, we have $(W^{(k)}, R^{(k)}, S^{(k)})^T \in \mathcal{S}(\overline{D}_\delta)$ for each $k \ge 1$. Thus it suggests that for $i = 1, \dots, 6$

$$
\begin{aligned}
& \left|C_1^{(k)}(t, z_0^{(k)}(t))\right|; \left|A_i^{(k)}(t, z_+^{(k)}(t))\right|; \left|B_i^{(k)}(t, z_-^{(k)}(t))\right| \le M, \\
& \left|C_{1z}^{(k)}\right|; t\left|C_{1R}^{(k)}\right|; t\left|C_{1S}^{(k)}\right|; \left|A_{iz}^{(k)}\right|; \left|A_{iS}^{(k)}\right|; \left|A_{iW}^{(k)}\right|; \left|B_{iz}^{(k)}\right|; \left|B_{iR}^{(k)}\right|; \left|B_{iW}^{(k)}\right| \le M
\end{aligned} \qquad (3.67)
$$

for any $(t,z) \in \overline{D}_\delta$. We next consider the estimates for $(W_\eta^{(k)}, R_\eta^{(k)}, S_\eta^{(k)})(\xi,\eta)$ in $\overline{D}_\delta$. Thanks to (3.48) and (3.25), we obtain for $k \ge 1$

(3.68)

$$
\left\{
\begin{aligned}
& W_\eta^{(k+1)}(\xi,\eta) \\
& = g_1(\xi_0^{(k)}) + \int_{\xi_0^{(k)}}^{\xi} \left\{ C_{11}^{(k)}tR_z^{(k)} + C_{12}^{(k)}tS_z^{(k)} + C_{13}^{(k)}t \right\} \frac{\partial z_0^{(k)}}{\partial \eta}(t, z_0^{(k)}(t))\, \mathrm{d}t, \\
& R_\eta^{(k+1)}(\xi,\eta) = g_2(\xi_+^{(k)}) + \int_{\xi_+^{(k)}}^{\xi} \left\{ \frac{R_z^{(k)} - S_z^{(k)}}{2t} + 2A_1^{(k)}\frac{(R^{(k)} - S^{(k)})(R_z^{(k)} - S_z^{(k)})}{t} \right. \\
& \qquad \left. + A_{11}^{(k)}R_z^{(k)} + A_{12}^{(k)}S_z^{(k)} + A_{13}^{(k)}W_z^{(k)} + A_{14}^{(k)} + T^{(k)}t \right\} \frac{\partial z_+^{(k)}}{\partial \eta}(t, z_+^{(k)}(t))\, \mathrm{d}t, \\
& S_\eta^{(k+1)}(\xi,\eta) = \int_0^{\xi} \left\{ \frac{S_z^{(k)} - R_z^{(k)}}{2t} + 2B_1^{(k)}\frac{(R^{(k)} - S^{(k)})(R_z^{(k)} - S_z^{(k)})}{t} + B_{11}^{(k)}S_z^{(k)} \right. \\
& \qquad \left. + B_{12}^{(k)}R_z^{(k)} + B_{13}^{(k)}W_z^{(k)} + B_{14}^{(k)} + T^{(k)}t \right\} \frac{\partial z_-^{(k)}}{\partial \eta}(t, z_-^{(k)}(t))\, \mathrm{d}t,
\end{aligned}
\right.
$$

where

$$
C_{11}^{(k)} = C_{1R}^{(k)}, \quad C_{12}^{(k)} = C_{1S}^{(k)}, \quad C_{13}^{(k)} = C_{1z}^{(k)}, \qquad (3.69)
$$

(3.70)
$$
\begin{aligned}
A_{11}^{(k)} &= A_2^{(k)}, \quad T^{(k)} = A_{6z}^{(k)} = B_{6z}^{(k)},\\
A_{12}^{(k)} &= A_3^{(k)} + A_{1S}^{(k)}\frac{(R^{(k)}-S^{(k)})^2}{t} + A_{2S}^{(k)}R^{(k)} + A_{3S}^{(k)}S^{(k)} + A_{4S}^{(k)}tW^{(k)} + A_{5S}^{(k)}t^2,\\
A_{13}^{(k)} &= tA_4^{(k)} + A_{1W}^{(k)}\frac{(R^{(k)}-S^{(k)})^2}{t} + A_{2W}^{(k)}R^{(k)} + A_{3W}^{(k)}S^{(k)} + A_{4W}^{(k)}tW^{(k)} + A_{5W}^{(k)}t^2,\\
A_{14}^{(k)} &= A_{1z}^{(k)}\frac{(R^{(k)}-S^{(k)})^2}{t} + A_{2z}^{(k)}R^{(k)} + A_{3z}^{(k)}S^{(k)} + A_{4z}^{(k)}tW^{(k)} + A_{5z}^{(k)}t^2,
\end{aligned}
$$

(3.71)
$$
\begin{aligned}
B_{11}^{(k)} &= B_3^{(k)},\\
B_{12}^{(k)} &= B_2^{(k)} + B_{1R}^{(k)}\frac{(R^{(k)}-S^{(k)})^2}{t} + B_{2R}^{(k)}R^{(k)} + B_{3R}^{(k)}S^{(k)} + B_{4R}^{(k)}tW^{(k)} + B_{5R}^{(k)}t^2,\\
B_{13}^{(k)} &= tB_4^{(k)} + B_{1W}^{(k)}\frac{(R^{(k)}-S^{(k)})^2}{t} + B_{2W}^{(k)}R^{(k)} + B_{3W}^{(k)}S^{(k)} + B_{4W}^{(k)}tW^{(k)} + B_{5W}^{(k)}t^2,\\
B_{14}^{(k)} &= B_{1z}^{(k)}\frac{(R^{(k)}-S^{(k)})^2}{t} + B_{2z}^{(k)}R^{(k)} + B_{3z}^{(k)}S^{(k)} + B_{4z}^{(k)}tW^{(k)} + B_{5z}^{(k)}t^2,
\end{aligned}
$$

and

$$
\frac{\partial z_i^{(k)}}{\partial \eta}(t;\xi,\eta) = \exp\left\{\int_\xi^t \frac{\partial \lambda_i^{(k)}}{\partial z}(t, z_i^{(k)}(t;\xi,\eta))\,\mathrm{d}t\right\}, \quad i=0,\pm. \tag{3.72}
$$

Furthermore, we recall the expressions of λ_i $(i=0,\pm)$ in (3.29) to see that

$$
\frac{\partial \lambda_0^{(k)}}{\partial z} = C_{14}^{(k)}R_z^{(k)} + C_{15}^{(k)}S_z^{(k)} + C_{16}^{(k)}, \tag{3.73}
$$

where

$$
\begin{aligned}
C_{14}^{(k)} &= \frac{\sqrt{1-t^2}t^2}{F[2a_1t-(R^{(k)}-S^{(k)})]} + \frac{\sqrt{1-t^2}(R^{(k)}+S^{(k)}+2a_0)t^2}{F[2a_1t-(R^{(k)}-S^{(k)})]^2},\\
C_{15}^{(k)} &= \frac{\sqrt{1-t^2}t^2}{F[2a_1t-(R^{(k)}-S^{(k)})]} - \frac{\sqrt{1-t^2}(R^{(k)}+S^{(k)}+2a_0)t^2}{F[2a_1t-(R^{(k)}-S^{(k)})]^2},\\
C_{16}^{(k)} &= \frac{2a_0'\sqrt{1-t^2}t^2}{F[2a_1t-(R^{(k)}-S^{(k)})]} - \frac{2a_1'\sqrt{1-t^2}(R^{(k)}+S^{(k)}+2a_0)t^3}{F[2a_1t-(R^{(k)}-S^{(k)})]^2},
\end{aligned} \tag{3.74}
$$

$$
\frac{\partial \lambda_+^{(k)}}{\partial z} = A_{15}^{(k)}S_z^{(k)} + A_{16}^{(k)}W_z^{(k)} + A_{17}^{(k)}, \tag{3.75}
$$

where

$$
\begin{aligned}
A_{15}^{(k)} &= \frac{\sqrt{1-t^2}t^2}{F[S^{(k)}+G_1W^{(k)}+\Psi]} - \frac{\sqrt{1-t^2}(S^{(k)}+a_0+a_1t)}{F[S^{(k)}+G_1W^{(k)}+\Psi]^2}t^2,\\
A_{16}^{(k)} &= -\frac{\sqrt{1-t^2}(S^{(k)}+a_0+a_1t)G_1}{F[S^{(k)}+G_1W^{(k)}+\Psi]^2}t^2,\\
A_{17}^{(k)} &= \frac{\sqrt{1-t^2}(a_0'+a_1't)t^2}{F[S^{(k)}+G_1W^{(k)}+\Psi]} - \frac{\sqrt{1-t^2}(S^{(k)}+a_0+a_1t)\Psi_z}{F[S^{(k)}+G_1W^{(k)}+\Psi]^2}t^2,
\end{aligned} \tag{3.76}
$$

and

$$\frac{\partial \lambda_-^{(k)}}{\partial z} = B_{15}^{(k)} R_z^{(k)} + B_{16}^{(k)} W_z^{(k)} + B_{17}^{(k)}, \tag{3.77}$$

where

$$\begin{aligned}
B_{15}^{(k)} &= -\frac{\sqrt{1-t^2}t^2}{F[R^{(k)} + G_1 W^{(k)} + \Phi]} + \frac{\sqrt{1-t^2}(R^{(k)} + a_0 - a_1 t)}{F[R^{(k)} + G_1 W^{(k)} + \Phi]^2} t^2, \\
B_{16}^{(k)} &= \frac{\sqrt{1-t^2}(R^{(k)} + a_0 - a_1 t) G_1}{F[R^{(k)} + G_1 W^{(k)} + \Phi]^2} t^2, \\
B_{17}^{(k)} &= -\frac{\sqrt{1-t^2}(a_0' - a_1' t)t^2}{F[R^{(k)} + G_1 W^{(k)} + \Phi]} + \frac{\sqrt{1-t^2}(R^{(k)} + a_0 - a_1 t)\Phi_z}{F[R^{(k)} + G_1 W^{(k)} + \Phi]^2} t^2.
\end{aligned} \tag{3.78}$$

Combining with (3.69)–(3.78) and using (3.67), (3.52), and the regularity conditions in (3.24), we can deduce the estimates

$$t\left|C_{11}^{(k)}\right| \le M,\ t\left|C_{12}^{(k)}\right| \le M,\ \left|C_{13}^{(k)}\right| \le M,\ \left|T^{(k)}\right| \le M,\ \left|T_z^{(k)}\right| \le M, \tag{3.79}$$

$$\begin{aligned}
&\left|f_{11}^{(k)}\right| \le M, \quad \left|f_{12}^{(k)}\right| \le M + M\cdot(3M)^2\delta^3 + 3M\cdot 3M\delta^2 + M\delta^2 \le 2M, \\
&\left|f_{13}^{(k)}\right| \le t[M + M\cdot(3M)^2\delta^2 + 3M\cdot 3M\delta + M\delta] \\
&\qquad \le Mt(1 + 9M^2\delta^2 + 10M\delta) \le 2Mt, \\
&\left|f_{14}^{(k)}\right| \le M\cdot(3M)^2 t^3 + 3M\cdot Mt^2 + Mt^2 \le Mt^2(10M + 9M^2\delta) \le 12M^2t^2,
\end{aligned} \tag{3.80}$$

where $f = A, B$, and

$$\begin{aligned}
&\left|C_{14}^{(k)}\right| \le M,\ \left|C_{15}^{(k)}\right| \le M,\ \left|C_{16}^{(k)}\right| \le Mt, \\
&\left|A_{15}^{(k)}\right|; \left|A_{16}^{(k)}\right|; \left|A_{17}^{(k)}\right|; \left|B_{15}^{(k)}\right|; \left|B_{16}^{(k)}\right|; \left|B_{17}^{(k)}\right| \le Mt^2,
\end{aligned} \tag{3.81}$$

and

$$\begin{aligned}
&\left|\frac{\partial z_0^{(k)}}{\partial \eta}\right| \le \exp\left\{ \int_0^\delta \left[M(\left|R_z^{(k)}\right| + \left|S_z^{(k)}\right|) + Mt\right]\, \mathrm{d}t \right\}, \\
&\left|\frac{\partial z_\pm^{(k)}}{\partial \eta}\right| \le \exp\left\{ \int_0^\delta Mt^2\left(\left|R_z^{(k)}\right| + \left|S_z^{(k)}\right| + \left|W_z^{(k)}\right| + 1\right)\, \mathrm{d}t \right\}.
\end{aligned} \tag{3.82}$$

Then we have the following lemma.

LEMMA 3.3. *For all $k \ge 1$ the inequalities*

$$\begin{aligned}
&\left|W_\eta^{(k)}(\xi,\eta)\right| \le M\xi \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j, \quad \left|R_\eta^{(k)}(\xi,\eta)\right|;\ \left|S_\eta^{(k)}(\xi,\eta)\right| \le M\xi^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j, \\
&\left|R_\eta^{(k)}(\xi,\eta) - S_\eta^{(k)}(\xi,\eta)\right| \le M\xi^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j
\end{aligned} \tag{3.83}$$

hold in $\overline{D}_\delta$.

Proof. We also apply the argument of induction to show the lemma. Due to the fact $W_z^{(0)} = R_z^{(0)} = S_z^{(0)} = 0$, one has by (3.82)

$$\left|\frac{\partial z_i^{(0)}}{\partial \eta}\right| \le \exp(M\delta^2), \quad i = 0, \pm, \tag{3.84}$$

which along with (3.68), (3.79), (3.80), and (3.26) get

$$\begin{aligned}\left|W_\eta^{(1)}(\xi,\eta)\right| &\le \left|g_1(\xi_0^{(0)})\right| + \int_0^\xi \left|C_{13}^{(0)}\right| \cdot t \cdot \left|\frac{\partial z_0^{(0)}}{\partial \eta}\right| \,\mathrm{d}t \\ &\le \widehat{K}\xi_0^{(0)} + \int_0^\xi Mt\exp(M\delta^2)\,\mathrm{d}t \\ &\le M\xi\left(\frac{1}{4} + \delta\exp(M\delta^2)\right) \le M\xi\sum_{j=0}^{1}\left(\frac{2}{3}\right)^j \end{aligned}\tag{3.85}$$

and

$$\begin{aligned}\left|R_\eta^{(1)}(\xi,\eta)\right| &\le \left|g_2(\xi_+^{(0)})\right| + \int_0^\xi \left(\left|A_{14}^{(0)}\right| + \left|T^{(0)}\right|t\right) \cdot \left|\frac{\partial z_+^{(0)}}{\partial \eta}\right| \,\mathrm{d}t \\ &\le \widehat{K}(\xi_+^{(0)})^2 + \int_0^\xi (12M^2t^2 + Mt)\exp(M\delta^2)\,\mathrm{d}t \\ &\le M\xi^2\left\{\left(\frac{1}{4} + \frac{1}{2}\exp(M\delta^2)\right) + 4M\delta\exp(M\delta^2)\right\} \le M\xi^2\sum_{j=0}^{1}\left(\frac{2}{3}\right)^j. \end{aligned}\tag{3.86}$$

Moreover, we have by (3.68)

$$\left|R_\eta^{(1)}(\xi,\eta) - S_\eta^{(1)}(\xi,\eta)\right| \le \left|g_2(\xi_+^{(0)})\right| + I_5 + I_6 + I_7, \tag{3.87}$$

where

$$\begin{aligned} I_5 &= \int_0^{\xi_+^{(0)}} \left(\left|B_{14}^{(0)}\right| + \left|T^{(0)}\right|t\right) \cdot \left|\frac{\partial z_-^{(0)}}{\partial \eta}\right| \,\mathrm{d}t, \\ I_6 &= \int_{\xi_+^{(0)}}^\xi \left\{\left|A_{14}^{(0)}\right| \cdot \left|\frac{\partial z_+^{(0)}}{\partial \eta}\right| + \left|B_{14}^{(0)}\right| \cdot \left|\frac{\partial z_-^{(0)}}{\partial \eta}\right|\right\} \mathrm{d}t, \\ I_7 &= \int_{\xi_+^{(0)}}^\xi \left|T^{(0)}\frac{\partial z_+^{(0)}}{\partial \eta}(t, z_+^{(0)}) - T^{(0)}\frac{\partial z_-^{(0)}}{\partial \eta}(t, z_-^{(0)})\right| \cdot t\,\mathrm{d}t. \end{aligned}$$

For I_5 and I_6, one uses (3.80) and (3.84) again to obtain

$$I_5 \le \int_0^\xi \left(12M^2t^2 + Mt\right)\exp(M\delta^2)\,\mathrm{d}t \le M\xi^2\left(\frac{1}{2} + 4M\delta\right)\exp(M\delta^2) \tag{3.88}$$

and

$$I_6 \le \int_0^\xi 2\cdot 12M^2t^2\exp(M\delta^2)\,\mathrm{d}t \le M\xi^2 \cdot 8M\delta\exp(M\delta^2). \tag{3.89}$$

For the term I_7, we also have by (3.72), (3.79), and (3.82)

$$\begin{aligned} I_7 \le \int_0^\xi \bigg\{ & \left|T^{(0)}(t, z_+^{(0)}(t)) - T^{(0)}(t, z_-^{(0)}(t))\right| \cdot \left|\frac{\partial z_+^{(0)}}{\partial \eta}\right| \\ & + \left|T^{(0)}(t, z_-^{(0)}(t))\right| \cdot \left|\frac{\partial z_+^{(0)}}{\partial \eta}(t, z_+^{(0)}(t)) - \frac{\partial z_-^{(0)}}{\partial \eta}(t, z_-^{(0)}(t))\right|\bigg\} \,\mathrm{d}t \\ \le \int_0^\xi \bigg\{ & M\left|z_+^{(0)}(t) - z_-^{(0)}(t)\right| \exp(M\delta^2) \\ & + M\exp(M\delta^2)\int_0^\xi \left(\left|\frac{\partial \lambda_+^{(0)}}{\partial z}\right| + \left|\frac{\partial \lambda_-^{(0)}}{\partial z}\right|\right) \mathrm{d}\tau \bigg\} \,\mathrm{d}t, \end{aligned}$$

which along with (3.43), (3.75), (3.77), and (3.81) gives

$$\begin{aligned} I_7 &\le \int_0^\xi \left\{ M^2\xi^3\exp(M\delta^2) + M\exp(M\delta^2)\int_0^\xi 2M\tau^2 \,\mathrm{d}\tau \right\} \mathrm{d}t \\ &\le 2M^2\xi^4\exp(M\delta^2). \end{aligned} \tag{3.90}$$

Putting (3.88)–(3.90) into (3.87) and employing (3.26) arrives at

$$\begin{aligned} \left|R_\eta^{(1)}(\xi,\eta) - S_\eta^{(1)}(\xi,\eta)\right| &\le \widehat{K}(\xi_+^{(0)})^2 + M\xi^2\left(\frac{1}{2} + 4M\delta\right)\exp(M\delta^2) \\ &\quad + M\xi^2 \cdot 8M\delta\exp(M\delta^2) + 2M^2\xi^4\exp(M\delta^2) \\ &\le M\xi^2\left\{\frac{1}{4} + \left(\frac{1}{2} + 13M\delta\right)\exp(M\delta^2)\right\} \le M\xi^2\sum_{j=0}^{1}\left(\frac{2}{3}\right)^j. \end{aligned} \tag{3.91}$$

Combining with (3.85)–(3.86) and (3.91) it follows that (3.83) holds for $n = 1$.

Now assume that (3.83) are true for $n = k$. Hence we see that

$$\left|W_\eta^{(k)}(\xi,\eta)\right| \le 3M\xi, \quad \left|R_\eta^{(k)}(\xi,\eta)\right|; \left|S_\eta^{(k)}(\xi,\eta)\right| \le 3M\xi^2,$$

from which and (3.82) we acquire

$$\left|\frac{\partial z_i^{(k)}}{\partial \eta}\right| \le \exp(2M\delta^2), \quad i = 0, \pm. \tag{3.92}$$

For $n = k+1$, we now apply (3.68), (3.79), (3.92), and (3.43) and the induction assumptions to obtain

$$\begin{aligned} & \left|W_\eta^{(k+1)}(\xi,\eta)\right| \\ & \le \left|g_1(\xi_0^{(k)})\right| + \int_0^\xi \left\{ t\left|C_{11}^{(k)}\right| \cdot \left|R_z^{(k)}\right| + t\left|C_{12}^{(k)}\right| \cdot \left|S_z^{(k)}\right| + t\left|C_{13}^{(k)}\right| \right\} \left|\frac{\partial z_0^{(k)}}{\partial \eta}\right| \mathrm{d}t \\ & \le \widehat{K}\xi_0^{(k)} + \int_0^\xi \left\{ 2M \cdot Mt^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + Mt \right\} \exp(2M\delta^2) \,\mathrm{d}t \\ & \le M\xi\left\{ \left(\frac{1}{4} + \delta\exp(2M\delta^2)\right) + M\delta^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j \right\} \le M\xi\sum_{j=0}^{k+1}\left(\frac{2}{3}\right)^j. \end{aligned} \tag{3.93}$$

A similar argument for $R_\eta^{(k)}$ leads to

$$
\begin{aligned}
&\left|R_\eta^{(k+1)}(\xi,\eta)\right| \\
&\leq \left|g_2(\xi_+^{(k)})\right| + \int_{\xi_+^{(k)}}^{\xi} \left\{ \frac{\left|R_z^{(k)} - S_z^{(k)}\right|}{2t} + 2M\cdot 3Mt\cdot\left|R_z^{(k)} - S_z^{(k)}\right| \right. \\
&\quad \left. + M\left|R_z^{(k)}\right| + 2M\cdot\left|S_z^{(k)}\right| + 2Mt\cdot\left|W_z^{(k)}\right| + 12M^2t^2 + Mt \right\} \left|\frac{\partial z_+^{(k)}}{\partial\eta}\right| \,\mathrm{d}t \\
&\leq \frac{1}{4}M\xi^2 + \int_0^{\xi} \left\{ \frac{1}{2}Mt\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + 6M^2t\cdot Mt^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j \right. \\
&\quad \left. + 5M^2t^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + 12M^2t^2 + Mt \right\} \exp(2M\delta^2)\,\mathrm{d}t \\
&\leq M\xi^2 \left\{ \left(\frac{1}{4} + (\frac{1}{2} + 4M\delta)\exp(2M\delta^2)\right) \right. \\
&\quad \left. + \left(\frac{1}{4} + 4M\delta\right)\exp(2M\delta^2)\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j \right\} \leq M\xi^2\sum_{j=0}^{k+1}\left(\frac{2}{3}\right)^j.
\end{aligned}
\tag{3.94}
$$

For the term $|R_\eta^{(k)}(\xi,\eta) - S_\eta^{(k)}(\xi,\eta)|$, we proceed by (3.68),

$$
\left|R_\eta^{(k+1)}(\xi,\eta) - S_\eta^{(k+1)}(\xi,\eta)\right| \leq \left|g_2(\xi_+^{(k)})\right| + I_8 + I_9 + I_{10} + I_{11},
\tag{3.95}
$$

where

$$
\begin{aligned}
I_8 &= \int_0^{\xi} \left\{ \frac{\left|R_z^{(k)} - S_z^{(k)}\right|}{2t} + 2\left|A_1^{(k)}\right| \frac{\left|R^{(k)} - S^{(k)}\right|\cdot\left|R_z^{(k)} - S_z^{(k)}\right|}{t} \right. \\
&\quad \left. + \left|A_{11}^{(k)}\right|\cdot\left|R_z^{(k)}\right| + \left|A_{12}^{(k)}\right|\cdot\left|S_z^{(k)}\right| + \left|A_{13}^{(k)}\right|\cdot\left|W_z^{(k)}\right| + \left|A_{14}^{(k)}\right| \right\} \left|\frac{\partial z_+^{(k)}}{\partial\eta}\right| \,\mathrm{d}t, \\
I_9 &= \int_0^{\xi} \left\{ \frac{\left|S_z^{(k)} - R_z^{(k)}\right|}{2t} + 2\left|B_1^{(k)}\right| \frac{\left|R^{(k)} - S^{(k)}\right|\cdot\left|R_z^{(k)} - S_z^{(k)}\right|}{t} + \left|B_{11}^{(k)}\right|\cdot\left|S_z^{(k)}\right| \right. \\
&\quad \left. + \left|B_{12}^{(k)}\right|\cdot\left|R_z^{(k)}\right| + \left|B_{13}^{(k)}\right|\cdot\left|W_z^{(k)}\right| + \left|B_{14}^{(k)}\right| \right\} \left|\frac{\partial z_-^{(k)}}{\partial\eta}\right| \,\mathrm{d}t, \\
I_{10} &= \int_0^{\xi_+^{(k)}} \left|T^{(k)}\right|\cdot\left|\frac{\partial z_-^{(k)}}{\partial\eta}(t, z_-^{(k)}(t))\right|\cdot t\,\mathrm{d}t, \\
I_{11} &= \int_{\xi_+^{(k)}}^{\xi} \left\{ \left| T^{(k)}\frac{\partial z_+^{(k)}}{\partial\eta}(t, z_+^{(k)}(t)) - T^{(k)}\frac{\partial z_-^{(k)}}{\partial\eta}(t, z_-^{(k)}(t)) \right| \cdot t\,\mathrm{d}t. \right.
\end{aligned}
$$

Making use of (3.52), (3.80), and (3.92), we find by the induction assumptions that

$$
\begin{aligned}
I_8; I_9 \le\ & \int_0^{\xi} \left\{ \frac{1}{2}Mt \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j + 2M\cdot 3Mt \cdot Mt^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j + M\cdot Mt^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \right. \\
& \left. + 2M\cdot Mt^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j + 2Mt\cdot Mt \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j + 12M^2t^2 \right\} \exp(2M\delta^2)\ \mathrm{d}t \\
\le\ & M\xi^2 \left(\frac{1}{4} + 4M\delta\right) \exp(2M\delta^2) \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j + 4M\xi^3 \exp(2M\delta^2). \qquad (3.96)
\end{aligned}
$$

From (3.79) and (3.92), it is easy to see that

$$
I_{10} \le \int_0^{\xi_+^{(k)}} Mt \exp(2M\delta^2)\ \mathrm{d}t \le \frac{1}{2} M(\xi_+^{(k)})^2 \exp(2M\delta^2). \qquad (3.97)
$$

For the term I_{11}, we have by (3.79), (3.92), and (3.72)

$$
\begin{aligned}
I_{11} \le\ & \int_{\xi_+^{(k)}}^{\xi} \left\{ \left|T^{(k)}(t, z_+^{(k)}(t)) - T^{(k)}(t, z_-^{(k)}(t))\right| \cdot \left|\frac{\partial z_+^{(k)}}{\partial \eta}\right| \right. \\
& \left. + \left|T^{(k)}(t, z_-^{(k)}(t))\right| \cdot \left| \frac{\partial z_+^{(k)}}{\partial \eta}(t, z_+^{(k)}(t)) - \frac{\partial z_-^{(k)}}{\partial \eta}(t, z_-^{(k)}(t)) \right| \right\} \cdot t\ \mathrm{d}t \\
\le\ & \int_{\xi_+^{(k)}}^{\xi} \left\{ M\left|z_+^{(k)}(t) - z_-^{(k)}(t)\right| \exp(2M\delta^2) \right. \\
& \left. + M\exp(2M\delta^2) \int_0^{\delta} \left(\left|\frac{\partial \lambda_+^{(k)}}{\partial z}\right| + \left|\frac{\partial \lambda_-^{(k)}}{\partial z}\right| \right) \mathrm{d}\tau \right\} \cdot t\ \mathrm{d}t,
\end{aligned}
$$

which along with (3.43), (3.75), (3.77), and (3.81) gets

$$
\begin{aligned}
I_{11} \le\ & \int_0^{\xi} \left\{ M^2\xi^3 \exp(2M\delta^2) + M\exp(2M\delta^2) \int_0^{\delta} 2\left(6M^2\tau^3 + M\tau^2\right) \mathrm{d}\tau \right\} \cdot t\ \mathrm{d}t \\
\le\ & M\xi^2 \exp(2M\delta^2) \left(2M\delta^3 + 3M^2\delta^4\right). \qquad (3.98)
\end{aligned}
$$

One inserts (3.96)–(3.98) into (3.95) and applies (3.26) to finally acquire

$$
\begin{aligned}
& \left|R_\eta^{(k+1)}(\xi,\eta) - S_\eta^{(k+1)}(\xi,\eta)\right| \\
& \le M\xi^2 \left\{ \left[\frac{1}{4} + \left(\frac{1}{2} + 8\delta + 2M\delta^3 + 3M^2\delta^4 \right) \exp(2M\delta^2) \right] \right. \\
& \qquad \left. + \left(\frac{1}{2} + 8M\delta\right) \exp(2M\delta^2) \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \right\} \le M\xi^2 \sum_{j=0}^{k+1} \left(\frac{2}{3}\right)^j. \qquad (3.99)
\end{aligned}
$$

We combine (3.93), (3.94), and (3.99) to finish the proof of the lemma. ∎

In view of Lemmas 3.2 and 3.3, we have the following.

LEMMA 3.4. *For all $k \ge 0$ the inequalities*

$$
\left|I^{(k+1)}(\xi,\eta) - I^{(k)}(\xi,\eta)\right| \le M\xi^2 \left(\frac{2}{3}\right)^k, \quad I = W, R, S, \qquad (3.100)
$$

hold in $\overline{D}_\delta$.

Proof. The proof of the lemma is also based on the argument of induction. It is obvious by (3.54) and (3.55) that

$$\left|W^{(1)}(\xi,\eta)-W^{(0)}(\xi,\eta)\right|\le\frac{3}{4}M\xi^2+K_0\xi^2\le M\xi^2$$

and

$$\begin{aligned}&\left|R^{(1)}(\xi,\eta)-R^{(0)}(\xi,\eta)\right|\\&\le\left|b(\xi)\right|+\left|b(\xi_+^{(0)})\right|+\int_{\xi_+^{(0)}}^{\xi}\left\{K_0t+M^3t^3+4M^2t^2+Mt\right\}\,\mathrm{d}t\\&\le\frac14M\xi^2+\frac14M(\xi_+^{(0)})^2+\frac{K_0}{2}\xi^2+M^3\xi^4+2M^2\xi^3+\frac12M\xi^2-\frac12M(\xi_+^{(0)})^2\\&\le M\xi^2\left(\frac78+M^2\delta^2+2M\delta\right)\le M\xi^2,\end{aligned}$$

which indicate that each of the inequalities in (3.100) holds for $n=0$. Assume they are valid for $n=k-1$. We shall check that they are preserved for $n=k$.

In order to achieve the goal, we first estimate the quantities $|\xi_i^{(k)}-\xi_i^{(k-1)}|$ $(i=0,+)$. Without loss of generality, we assume $\xi_i^{(k)}\ge\xi_i^{(k-1)}$ $(i=0,+)$. Recalling (3.30), one knows that for $i=0,+$

$$\begin{aligned}&\int_{\xi_i^{(k)}}^{\xi}\lambda_i^{(k)}(t,z_i^{(k)}(t;\xi,\eta))\,\mathrm{d}t+\int_0^{\xi_i^{(k)}}\lambda_-(t,\tilde z(t))\,\mathrm{d}t\\&=\eta=\int_{\xi_i^{(k-1)}}^{\xi}\lambda_i^{(k-1)}(t,z_i^{(k-1)}(t;\xi,\eta))\,\mathrm{d}t+\int_0^{\xi_i^{(k-1)}}\lambda_-(t,\tilde z(t))\,\mathrm{d}t,\end{aligned}$$

from which we deduce

$$\int_{\xi_i^{(k-1)}}^{\xi_i^{(k)}}(\lambda_i^{(k-1)}-\lambda_-)\,\mathrm{d}t=\int_{\xi_i^{(k)}}^{\xi}\left(\lambda_i^{(k)}(t,z_i^{(k)}(t))-\lambda_i^{(k-1)}(t,z_i^{(k-1)}(t))\right)\,\mathrm{d}t.$$

Thus for $i=0$, we have by (3.40) and (3.81)

$$\begin{aligned}&\frac12\underline{k}\left|(\xi_0^{(k)})^2-(\xi_0^{(k-1)})^2\right|\\&\le\int_{\xi_0^{(k)}}^{\xi}\Big(\left|\lambda_{0R}\right|\cdot\left|R^{(k)}(t,z_0^{(k)}(t))-R^{(k-1)}(t,z_0^{(k-1)}(t))\right|\\&\qquad+\left|\lambda_{0S}\right|\cdot\left|S^{(k)}(t,z_0^{(k)}(t))-S^{(k-1)}(t,z_0^{(k-1)}(t))\right|\\&\qquad+\left|\lambda_{0z}\right|\cdot\left|z_0^{(k)}(t)-z_0^{(k-1)}(t)\right|\Big)\,\mathrm{d}t\\&\le\int_{\xi_0^{(k)}}^{\xi}\left(\left|C_{14}\right|I_{12}+\left|C_{15}\right|I_{13}+\left|C_{16}\right|\cdot\left|z_1^{(k)}(t)-z_1^{(k-1)}(t)\right|\right)\,\mathrm{d}t\\&\le\int_{\xi_0^{(k)}}^{\xi}\left(M(I_{12}+I_{13})+Mt\left|z_0^{(k)}(t)-z_0^{(k-1)}(t)\right|\right)\,\mathrm{d}t,\end{aligned}\tag{3.101}$$

where

$$I_{12} = \big|R^{(k)}(t, z_0^{(k)}(t)) - R^{(k-1)}(t, z_0^{(k-1)}(t))\big|,$$
$$I_{13} = \big|S^{(k)}(t, z_0^{(k)}(t)) - S^{(k-1)}(t, z_0^{(k-1)}(t))\big|.$$

Similarly, for $i = +$, it follows that

$$\begin{aligned}
&\frac{1}{3}\underline{k}\big|(\xi_+^{(k)})^3 - (\xi_+^{(k-1)})^3\big| \\
&\leq \int_{\xi_+^{(k)}}^{\xi} \Big(\big|\lambda_{+S}\big| \cdot \big|S^{(k)}(t, z_+^{(k)}(t)) - S^{(k-1)}(t, z_+^{(k-1)}(t))\big| \\
&\qquad + \big|\lambda_{+W}\big| \cdot \big|W^{(k)}(t, z_+^{(k)}(t)) - W^{(k-1)}(t, z_+^{(k-1)}(t))\big| \\
&\qquad + \big|\lambda_{+z}\big| \cdot \big|z_+^{(k)}(t) - z_+^{(k-1)}(t)\big|\Big)\,\mathrm{d}t \\
&\leq \int_{\xi_+^{(k)}}^{\xi} \Big(\big|A_{15}\big|I_{14} + \big|A_{16}\big|I_{15} + \big|A_{17}\big| \cdot \big|z_+^{(k)}(t) - z_+^{(k-1)}(t)\big|\Big)\,\mathrm{d}t \\
&\leq \int_{\xi_+^{(k)}}^{\xi} Mt^2\Big(I_{14} + I_{15} + \big|z_+^{(k)}(t) - z_+^{(k-1)}(t)\big|\Big)\,\mathrm{d}t,
\end{aligned} \tag{3.102}$$

where

$$I_{14} = \big|S^{(k)}(t, z_+^{(k)}(t)) - S^{(k-1)}(t, z_+^{(k-1)}(t))\big|,$$
$$I_{15} = \big|W^{(k)}(t, z_+^{(k)}(t)) - W^{(k-1)}(t, z_+^{(k-1)}(t))\big|.$$

To estimate I_i $(i = 12, \ldots, 15)$, we apply (3.83) and the induction assumptions to find that

$$\begin{aligned}
&\big|I^{(k)}(t, z_i^{(k)}(t)) - I^{(k-1)}(t, z_i^{(k-1)}(t))\big| \\
&\leq \big|I^{(k)}(t, z_i^{(k)}(t)) - I^{(k)}(t, z_i^{(k-1)}(t))\big| \\
&\qquad + \big|I^{(k)}(t, z_i^{(k-1)}(t)) - I^{(k-1)}(t, z_i^{(k-1)}(t))\big| \\
&\leq \big|I_z^{(k)}\big| \cdot \big|z_i^{(k)}(t) - z_i^{(k-1)}(t)\big| + Mt^2\left(\frac{2}{3}\right)^{k-1} \\
&\leq 3Mt\big|z_i^{(k)}(t) - z_i^{(k-1)}(t)\big| + Mt^2\left(\frac{2}{3}\right)^{k-1}
\end{aligned} \tag{3.103}$$

for $I = W, R, S$ and $i = 0, \pm$. Thus one has

$$\begin{aligned}
I_{12}; I_{13} &\leq 3Mt\big|z_0^{(k)}(t) - z_0^{(k-1)}(t)\big| + Mt^2\left(\frac{2}{3}\right)^{k-1}, \\
I_{14}; I_{15} &\leq 3Mt\big|z_+^{(k)}(t) - z_+^{(k-1)}(t)\big| + Mt^2\left(\frac{2}{3}\right)^{k-1}.
\end{aligned} \tag{3.104}$$

In addition, we recall the relations for $i = 0, +$

$$\begin{aligned}
&z_i^{(k)}(t) + \int_t^{\xi} \lambda_i^{(k)}(\tau, z_i^{(k)}(\tau))\,\mathrm{d}\tau \\
&= \eta = z_i^{(k-1)}(t) + \int_t^{\xi} \lambda_i^{(k-1)}(\tau, z_i^{(k-1)}(\tau))\,\mathrm{d}\tau \quad \forall\ t \in [\xi_i^{(k)}, \xi]
\end{aligned}$$

to get as the derivations of (3.101) and (3.102)

$$\begin{aligned}\big|z_0^{(k)}(t)-z_0^{(k-1)}(t)\big| &\le \int_t^\xi \big|\lambda_0^{(k)}(\tau,z_0^{(k)}(\tau))-\lambda_0^{(k-1)}(\tau,z_0^{(k-1)}(\tau))\big|\,\mathrm{d}\tau\\ &\le \int_t^\xi \left(M(I_{12}+I_{13})+M\tau\big|z_0^{(k)}(\tau)-z_0^{(k-1)}(\tau)\big|\right)\mathrm{d}\tau \qquad (3.105)\end{aligned}$$

and

$$\begin{aligned}\big|z_+^{(k)}(t)-z_+^{(k-1)}(t)\big| &\le \int_t^\xi \big|\lambda_+^{(k)}(\tau,z_+^{(k)}(\tau))-\lambda_+^{(k-1)}(\tau,z_+^{(k-1)}(\tau))\big|\,\mathrm{d}\tau\\ &\le \int_t^\xi M\tau^2\left(I_{14}+I_{15}+\big|z_+^{(k)}(\tau)-z_+^{(k-1)}(\tau)\big|\right)\mathrm{d}\tau. \qquad (3.106)\end{aligned}$$

One puts (3.104) into (3.105) and (3.106) respectively to arrive at

$$\begin{aligned}&\big|z_0^{(k)}(t)-z_0^{(k-1)}(t)\big|\\ &\quad\le \int_t^\xi (6M^2\tau+M\tau)\big|z_0^{(k)}(\tau)-z_0^{(k-1)}(\tau)\big|\,\mathrm{d}\tau+M^2\xi^3\left(\frac{2}{3}\right)^{k-1} \qquad (3.107)\end{aligned}$$

for $t\in[\xi_0^{(k)},\xi]$ and

$$\begin{aligned}&\big|z_+^{(k)}(t)-z_+^{(k-1)}(t)\big|\\ &\quad\le \int_t^\xi (6M^2\tau^3+M\tau^2)\big|z_+^{(k)}(\tau)-z_+^{(k-1)}(\tau)\big|\,\mathrm{d}\tau+M^2\xi^5\left(\frac{2}{3}\right)^{k-1} \qquad (3.108)\end{aligned}$$

for $t\in[\xi_+^{(k)},\xi]$. Now set

$$d_i^{(k)}=\max_{t\in[\xi_i^{(k)},\xi]}\big|z_i^{(k)}(t)-z_i^{(k-1)}(t)\big|,\quad i=0,+.$$

Then we acquire by (3.107) and (3.108)

$$d_0^{(k)}\le 4M^2\delta^2 d_0^{(k)}+M^2\xi^3\left(\frac{2}{3}\right)^{k-1},\quad d_+^{(k)}\le 4M^2\delta^2 d_+^{(k)}+M^2\xi^5\left(\frac{2}{3}\right)^{k-1},$$

which mean by the fact $4M^2\delta^2\le\frac{1}{2}$ that

$$d_0^{(k)}\le 2M^2\xi^3\left(\frac{2}{3}\right)^{k-1},\quad d_+^{(k)}\le 2M^2\xi^5\left(\frac{2}{3}\right)^{k-1}. \qquad (3.109)$$

Combining with (3.101), (3.102), (3.104), and (3.109) yields

$$\begin{aligned}\big|(\xi_0^{(k)})^2-(\xi_0^{(k-1)})^2\big| &\le \frac{2}{\underline{k}}\int_{\xi_0^{(k)}}^\xi \left\{2M\left(3Mtd_0^{(k)}+Mt^2\left(\frac{2}{3}\right)^{k-1}\right)+Mtd_0^{(k)}\right\}\mathrm{d}t\\ &\le \frac{2}{\underline{k}}(7M^4\xi^5+M^2\xi^3)\left(\frac{2}{3}\right)^{k-1}\le\frac{4M^2\xi^3}{\underline{k}}\left(\frac{2}{3}\right)^{k-1} \qquad (3.110)\end{aligned}$$

and

$$
\begin{aligned}
\left|(\xi_+^{(k)})^3-(\xi_+^{(k-1)})^3\right| &\le \frac{3}{\underline{k}}\int_{\xi_+^{(k)}}^{\xi} Mt^2\left\{2\left(3Mtd_+^{(k)}+Mt^2\left(\frac{2}{3}\right)^{k-1}\right)+d_+^{(k)}\right\}\mathrm{d}t\\
&\le \frac{3}{\underline{k}}(2M^3\xi^8+M^2\xi^5)\left(\frac{2}{3}\right)^{k-1}\le \frac{6M^2\xi^5}{\underline{k}}\left(\frac{2}{3}\right)^{k-1}. \qquad (3.111)
\end{aligned}
$$

We now check that (3.100) are valid for $n=k$. For the function W, we see by (3.48) that

$$\left|W^{(k+1)}(\xi,\eta)-W^{(k)}(\xi,\eta)\right|\le I_{16}+I_{17}, \qquad (3.112)$$

where

$$
\begin{aligned}
I_{16}&=\left|h(\xi_0^{(k)})-h(\xi_0^{(k-1)})-\int_{\xi_0^{(k-1)}}^{\xi_0^{(k)}}C_1^{(k-1)}(t,z_0^{(k-1)}(t))\cdot t\ \mathrm{d}t\right|,\\
I_{17}&=\int_{\xi_0^{(k)}}^{\xi}\left|C_1^{(k)}(t,z_0^{(k)}(t))-C_1^{(k-1)}(t,z_0^{(k-1)}(t))\right|\cdot t\ \mathrm{d}t.
\end{aligned}
$$

For the term I_{16}, it follows that

$$
\begin{aligned}
I_{16}&\le \int_{\xi_0^{(k-1)}}^{\xi_0^{(k)}}\left|h'(t)-C_1^{(k-1)}(t,z_0^{(k-1)}(t))\cdot t\right|\ \mathrm{d}t\\
&\le \int_{\xi_0^{(k-1)}}^{\xi_0^{(k)}}\left|h'(t)-C_1(t,\tilde z(t))\cdot t\right|\ \mathrm{d}t\\
&\quad+\int_{\xi_0^{(k-1)}}^{\xi_0^{(k)}}\left|C_1(t,\tilde z(t))-C_1^{(k-1)}(t,z_0^{(k-1)}(t))\right|\cdot t\ \mathrm{d}t\\
&=: I_{18}+I_{19}. \qquad (3.113)
\end{aligned}
$$

We recall (3.16)–(3.17) and the expression of C_1 in (3.27) to know that

(3.114)
$$
\begin{aligned}
h'(t)-C_1(t,\tilde z(t))\cdot t&=\tilde h_0'(t)-h_0'(\tilde z(t))\cdot\tilde z'(t)-\lambda_0(t,\tilde z(t))h_0'(\tilde z(t))\\
&=[\lambda_-(t,\tilde z(t))-\lambda_0(t,\tilde z(t))]\tilde g_1(t)-h_0'(\tilde z(t))\lambda_-(t,\tilde z(t))+\lambda_0(t,\tilde z(t))\hat h_0'(\tilde r(t))\\
&=\lambda_0(t,\tilde z(t))\cdot[\hat h_0'(\tilde r(t))-\tilde g_1(t)]+\lambda_-(t,\tilde z(t))\cdot[\tilde g_1(t)-h_0'(\tilde z(t))],
\end{aligned}
$$

from which and the condition $\mathcal{C}_1$ in (3.19) one has

$$\left|h'(t)-C_1(t,\tilde z(t))\cdot t\right|\le K_0t^2. \qquad (3.115)$$

Thus we combine (3.115) and (3.110) to get

$$
\begin{aligned}
I_{18}&\le\frac{K_0}{3}\left|(\xi_0^{(k)})^3-(\xi_0^{(k-1)})^3\right|\le\frac{M}{12}\xi\left|(\xi_0^{(k)})^2-(\xi_0^{(k-1)})^2\right|\\
&\le M\xi^2\frac{M^2\delta^2}{\underline{k}}\left(\frac{2}{3}\right)^{k-1}. \qquad (3.116)
\end{aligned}
$$

We now derive the estimate of I_{19} by employing (3.38), (3.83), and (3.110),

$$
\begin{aligned}
I_{19} &\le \int_{\xi_0^{(k-1)}}^{\xi_0^{(k)}} \Big\{ \big|C_{1R}\big| \cdot \big|R(t,\tilde{z}(t)) - R^{(k-1)}(t, z_0^{(k-1)}(t))\big| \\
&\quad + \big|C_{1S}\big| \cdot \big|S(t,\tilde{z}(t)) - S^{(k-1)}(t, z_0^{(k-1)}(t))\big| + \big|C_{1z}\big| \cdot \big|\tilde{z}(t) - z_0^{(k-1)}(t)\big| \Big\} \cdot t \,\mathrm{d}t \\
&= \int_{\xi_0^{(k-1)}}^{\xi_0^{(k)}} \Big\{ \big|C_{1R}\big| \cdot \big|R^{(k-1)}(t,\tilde{z}(t)) - R^{(k-1)}(t, z_0^{(k-1)}(t))\big| \\
&\quad + \big|C_{1S}\big| \cdot \big|S^{(k-1)}(t,\tilde{z}(t)) - S^{(k-1)}(t, z_0^{(k-1)}(t))\big| + \big|C_{1z}\big| \cdot \big|\tilde{z}(t) - z_0^{(k-1)}(t)\big| \Big\} \cdot t \,\mathrm{d}t \\
&\le \int_{\xi_0^{(k-1)}}^{\xi_0^{(k)}} \Big\{ \big|C_{1R}\big| \cdot \big|R_z^{(k-1)}\big| + \big|C_{1S}\big| \cdot \big|S_z^{(k-1)}\big| + \big|C_{1z}\big| \Big\} \cdot (\big|\tilde{z}(t)\big| + \big|z_0^{(k-1)}(t)\big|) t \,\mathrm{d}t \\
&\le \int_{\xi_0^{(k-1)}}^{\xi_0^{(k)}} \Big\{ \frac{2M}{t} \cdot 3Mt^2 + M \Big\} \cdot 2K_0 t^2 \cdot t \,\mathrm{d}t \le \frac{M^2}{4} \big|(\xi_0^{(k)})^4 - (\xi_0^{(k-1)})^4\big| \\
&\le \frac{M^2}{4} \xi^2 \big|(\xi_0^{(k)})^2 - (\xi_0^{(k-1)})^2\big| \le M\xi^2 \frac{M^3\delta^3}{\underline{k}} \left(\frac{2}{3}\right)^{k-1}.
\end{aligned}
$$

For the term I_{17}, one obtains by the definitions of I_{12} and I_{13} in (3.101)

$$
\begin{aligned}
I_{17} &\le \int_0^{\xi} \left(\big|C_{1R}\big| I_{12} + \big|C_{1S}\big| I_{13} + \big|C_{1z}\big| d_0^{(k)} \right) \cdot t \,\mathrm{d}t \\
&\le \int_0^{\xi} \left((6M^2 t + Mt) d_0^{(k)} + 2M^2 t^2 \left(\frac{2}{3}\right)^{k-1} \right) \cdot t \,\mathrm{d}t \\
&\le M\xi^2 (7M^3\delta^4 + M\delta^2) \left(\frac{2}{3}\right)^{k-1} \le M\xi^2 \cdot \delta \cdot \left(\frac{2}{3}\right)^{k-1}. \qquad (3.117)
\end{aligned}
$$

Inserting (3.113) and (3.116)–(3.117) into (3.112) gives

$$
\begin{aligned}
\big|W^{(k+1)}(\xi,\eta) - W^{(k)}(\xi,\eta)\big| &\le M\xi^2 \left(\delta + \frac{M^2\delta^2}{\underline{k}} + \frac{M^3\delta^3}{\underline{k}} \right) \left(\frac{2}{3}\right)^{k-1} \\
&\le M\xi^2 \left(\frac{2}{3}\right)^{k} \qquad (3.118)
\end{aligned}
$$

by the choice of δ in (3.50).

We next check the inequality for the function R in (3.100). From (3.48), we have

$$
\begin{aligned}
&\big|R^{(k+1)}(\xi,\eta) - R^{(k)}(\xi,\eta)\big| \\
&\quad \le I_{20} + \int_{\xi_+^{(k)}}^{\xi} \Big\{ I_{21} + I_{22} + I_{23} + I_{24} + I_{25} + I_{26} + I_{27} \Big\} \,\mathrm{d}t, \qquad (3.119)
\end{aligned}
$$

where

$$
\begin{aligned}
I_{20} = \Big| & b(\xi_+^{(k)}) - b(\xi_+^{(k-1)}) \\
& - \int_{\xi_+^{(k-1)}}^{\xi_+^{(k)}} \Big\{ \frac{R^{(k-1)} - S^{(k-1)}}{2t} + A_1^{(k-1)} \frac{(R^{(k-1)} - S^{(k-1)})^2}{t} + A_2^{(k-1)} R^{(k-1)} \\
& + A_3^{(k-1)} S^{(k-1)} + A_4^{(k-1)} t W^{(k-1)} + A_5^{(k-1)} t^2 + A_6^{(k-1)} t \Big\} (t, z_+^{(k-1)}(t)) \, \mathrm{d}t \Big| \\
=: & \Big| b(\xi_+^{(k)}) - b(\xi_+^{(k-1)}) - \int_{\xi_+^{(k-1)}}^{\xi_+^{(k)}} \Theta^{(k-1)}(t, z_+^{(k-1)}(t)) \, \mathrm{d}t \Big|
\end{aligned}
$$

and

$$
\begin{aligned}
& I_{21} = \frac{\big| R^{(k)}(t, z_+^{(k)}(t)) - R^{(k-1)}(t, z_+^{(k-1)}(t)) \big| + \big| S^{(k)}(t, z_+^{(k)}(t)) - S^{(k-1)}(t, z_+^{(k-1)}(t)) \big|}{2t}, \\
& I_{22} = \Big| A_1^{(k)} \frac{(R^{(k)} - S^{(k)})^2}{t}(t, z_+^{(k)}(t)) - A_1^{(k-1)} \frac{(R^{(k-1)} - S^{(k-1)})^2}{t}(t, z_+^{(k-1)}(t)) \Big|, \\
& I_{23} = \big| A_2^{(k)} R^{(k)}(t, z_+^{(k)}(t)) - A_2^{(k-1)} R^{(k-1)}(t, z_+^{(k-1)}(t)) \big|, \\
& I_{24} = \big| A_3^{(k)} S^{(k)}(t, z_+^{(k)}(t)) - A_3^{(k-1)} S^{(k-1)}(t, z_+^{(k-1)}(t)) \big|, \\
& I_{25} = t \cdot \big| A_4^{(k)} W^{(k)}(t, z_+^{(k)}(t)) - A_4^{(k-1)} W^{(k-1)}(t, z_+^{(k-1)}(t)) \big|, \\
& I_{26} = t^2 \cdot \big| A_5^{(k)}(t, z_+^{(k)}(t)) - A_5^{(k-1)}(t, z_+^{(k-1)}(t)) \big|, \\
& I_{27} = t \cdot \big| A_6^{(k)}(t, z_+^{(k)}(t)) - A_6^{(k-1)}(t, z_+^{(k-1)}(t)) \big|.
\end{aligned}
$$

It follows as in (3.113) that

$$
\begin{aligned}
I_{20} & \le \int_{\xi_+^{(k-1)}}^{\xi_+^{(k)}} \big| b'(t) - \Theta^{(k-1)}(t, z_+^{(k-1)}(t)) \big| \, \mathrm{d}t \\
& \le \int_{\xi_+^{(k-1)}}^{\xi_+^{(k)}} \big| b'(t) - \Theta(t, \tilde{z}(t)) \big| \, \mathrm{d}t + \int_{\xi_+^{(k-1)}}^{\xi_+^{(k)}} \big| \Theta(t, \tilde{z}(t)) - \Theta^{(k-1)}(t, z_+^{(k-1)}(t)) \big| \, \mathrm{d}t.
\end{aligned} \tag{3.120}
$$

Applying the same argument as in (3.115), we can obtain by (3.18) and (3.19) that

$$
\big| b'(t) - \Theta(t, \tilde{z}(t)) \big| \le K_0 t^3. \tag{3.121}
$$

Similar discussions as I_{19} arrive at

$$
\begin{aligned}
& \big| \Theta(t, \tilde{z}(t)) - \Theta^{(k-1)}(t, z_+^{(k-1)}(t)) \big| = \big| \Theta^{(k-1)}(t, \tilde{z}(t)) - \Theta^{(k-1)}(t, z_+^{(k-1)}(t)) \big| \\
& \quad \le \Big\{ \frac{\big| R_z^{(k-1)} \big| + \big| S_z^{(k-1)} \big|}{2t} + \Big| \frac{\partial A_1^{(k-1)}}{\partial z} \Big| \frac{(R^{(k-1)} - S^{(k-1)})^2}{t} \\
& \qquad + \big| A_1^{(k-1)} \big| \frac{2 \big| R^{(k-1)} - S^{(k-1)} \big| (\big| R_z^{(k-1)} \big| + \big| R_z^{(k-1)} \big|)}{t} + \Big| \frac{\partial A_2^{(k-1)}}{\partial z} \Big| \big| R^{(k-1)} \big|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left|A_2^{(k-1)}\right| \cdot \left|R_z^{(k-1)}\right| + \left|\frac{\partial A_3^{(k-1)}}{\partial z}\right| \left|S^{(k-1)}\right| \\
&+ \left|A_3^{(k-1)}\right| \cdot \left|S_z^{(k-1)}\right| + \left|\frac{\partial A_4^{(k-1)}}{\partial z}\right| \left|W^{(k-1)}\right| t \\
&+ \left|A_4^{(k-1)}\right| \cdot \left|W_z^{(k-1)}\right| t + \left|\frac{\partial A_5^{(k-1)}}{\partial z}\right| t^2 + \left|\frac{\partial A_6^{(k-1)}}{\partial z}\right| t \Bigg\} \left(\left|\tilde{z}(t)\right| + \left|z_+^{(k-1)}(t)\right|\right) \\
\le& \left\{6Mt + 54M^3t^3 + 36M^2t^2\right\} \cdot 2K_0 t^3 \le 5M^2 t^4.
\end{aligned} \tag{3.122}
$$

Here we used (3.52), (3.67), (3.83) and the following fact:

$$
\begin{aligned}
\left|\frac{\partial A_i^{(k-1)}}{\partial z}\right| &\le \left|A_{iR}^{(k-1)}\right| \cdot \left|R_z^{(k-1)}\right| + \left|A_{iS}^{(k-1)}\right| \cdot \left|S_z^{(k-1)}\right| + \left|A_{iz}^{(k-1)}\right| \\
&\le M \cdot 3Mt^2 + M \cdot 3Mt^2 + M \le 6M^2\delta^2 + M \le 2M.
\end{aligned}
$$

We now put (3.121) and (3.122) into (3.120) and employ (3.111) to deduce

$$
\begin{aligned}
I_{20} &\le \frac{K_0}{4}\left|(\xi_+^{(k)})^4 - (\xi_+^{(k-1)})^4\right| + M^2\left|(\xi_+^{(k)})^5 - (\xi_+^{(k-1)})^5\right| \\
&\le (M\xi + M^2\xi^2)\left|(\xi_+^{(k)})^3 - (\xi_+^{(k-1)})^3\right| \le 2M\xi \cdot \frac{6M^2\xi^5}{\underline{k}}\left(\frac{2}{3}\right)^{k-1} \\
&\le M\xi^2 \frac{\delta^2}{\underline{k}}\left(\frac{2}{3}\right)^{k-1}.
\end{aligned} \tag{3.123}
$$

Moreover, we recall (3.103) and (3.109) to achieve the estimate of I_{21},

$$
I_{21} \le 3Md_+^{(k)} + Mt\left(\frac{2}{3}\right)^{k-1} \le (6M^3\xi^5 + Mt)\left(\frac{2}{3}\right)^{k-1}. \tag{3.124}
$$

For the term I_{22}, one has

$$
\begin{aligned}
I_{22} \le& \frac{(R^{(k)} - S^{(k)})^2}{t} \cdot \left|A_1^{(k)}(t, z_+^{(k)}(t)) - A_1^{(k-1)}(t, z_+^{(k-1)}(t))\right| \\
&+ \left|A_1^{(k-1)}\right| \frac{\left|(R^{(k)} - S^{(k)})^2(t, z_+^{(k)}(t)) - (R^{(k-1)} - S^{(k-1)})^2(t, z_+^{(k-1)}(t))\right|}{t} \\
\le& 9M^2t^3 \cdot \Big\{ M\left|S^{(k)}(t, z_+^{(k)}(t)) - S^{(k-1)}(t, z_+^{(k-1)}(t))\right| \\
&\quad + M\left|W^{(k)}(t, z_+^{(k)}(t)) - W^{(k-1)}(t, z_+^{(k-1)}(t))\right| + M\left|z_+^{(k)}(t) - z_+^{(k-1)}(t)\right| \Big\} \\
&+ M \cdot 6Mt \cdot \Big\{ \left|R^{(k)}(t, z_+^{(k)}(t)) - R^{(k-1)}(t, z_+^{(k-1)}(t))\right| \\
&\quad + \left|S^{(k)}(t, z_+^{(k)}(t)) - S^{(k-1)}(t, z_+^{(k-1)}(t))\right| \Big\},
\end{aligned}
$$

which together with (3.103) and (3.109) yields

$$
\begin{aligned}
I_{22} &\le 2(9M^3t^3+6M^2t)\left(3Mtd_+^{(k)}+Mt^2\left(\frac{2}{3}\right)^{k-1}\right)+9M^3t^3d_+^{(k)} \\
&\le Md_+^{(k)}+6(3Mt^2+2)M^3t^3\left(\frac{2}{3}\right)^{k-1}\le 2M^3\xi^5\left(\frac{2}{3}\right)^{k-1}+18M^3t^3\left(\frac{2}{3}\right)^{k-1}.
\end{aligned}
\tag{3.125}
$$

In a similar way as for I_{22}, we also have

$$
\begin{aligned}
I_{23};I_{24};I_{25} &\le Md_+^{(k)}+(M+3M^2t^2)Mt^2\left(\frac{2}{3}\right)^{k-1} \\
&\le 2M^3\xi^5\left(\frac{2}{3}\right)^{k-1}+2M^2t^2\left(\frac{2}{3}\right)^{k-1}
\end{aligned}
\tag{3.126}
$$

and

$$
\begin{aligned}
&I_{26}\le td_+^{(k)}+2M^2t^4\left(\frac{2}{3}\right)^{k-1}\le M^2\xi^5\left(\frac{2}{3}\right)^{k-1}+2M^2t^4\left(\frac{2}{3}\right)^{k-1}, \\
&I_{27}\le tMd_+^{(k)}\le M^2\xi^5\left(\frac{2}{3}\right)^{k-1}.
\end{aligned}
\tag{3.127}
$$

Summing up (3.119) and (3.123)–(3.127), we finally arrive at

$$
\begin{aligned}
&\left|R^{(k+1)}(\xi,\eta)-R^{(k)}(\xi,\eta)\right| \\
&\le M\xi^2\frac{\delta^2}{\underline{k}}\left(\frac{2}{3}\right)^{k-1}+\left(\frac{2}{3}\right)^{k-1}\int_0^\xi\Big\{(6M^3\xi^5+Mt)+(2M^3\xi^5+18M^3t^3) \\
&\quad+3(2M^3\xi^5+2M^2t^2)+2(M^2\xi^5+2M^2t^4)\Big\}\,\mathrm{d}t \\
&\le M\xi^2\left(\frac{\delta^2}{\underline{k}}+\frac{1}{2}+4M\delta\right)\left(\frac{2}{3}\right)^{k-1}\le M\xi^2\left(\frac{2}{3}\right)^{k}
\end{aligned}
\tag{3.128}
$$

by the choice of δ in (3.50). The above estimate is also true for S. The proof of the lemma is complete by combining (3.118) and (3.128). ∎

We here provide a further remark on the higher-order compatibility condition $(\mathcal{C}_2)$ in (3.19).

Remark 2. The condition $(\mathcal{C}_2)$ in (3.19) is mainly used to establish the estimate $|g_2(\xi_+^{(k)})|\le K_0\xi^2$ in the proof of Lemma 3.3. We comment that this condition $(\mathcal{C}_2)$ and the corresponding regularity conditions in (2.19) and (2.21) can be relaxed such that the function $g_2(t)$ satisfies $|g_2(t)|\le K_0t^{1+\nu}$ for constant $\nu>0$. If so, then we have the following inequalities instead of (3.83) in Lemma 3.3:

$$
\begin{aligned}
&\left|W_\eta^{(k)}(\xi,\eta)\right|\le M\xi\sum_{j=0}^k\left(\frac{2}{3}\right)^j,\quad \left|R_\eta^{(k)}(\xi,\eta)\right|;\left|S_\eta^{(k)}(\xi,\eta)\right|\le M\xi^{1+\nu}\sum_{j=0}^k\left(\frac{2}{3}\right)^j, \\
&\left|R_\eta^{(k)}(\xi,\eta)-S_\eta^{(k)}(\xi,\eta)\right|\le M\xi^{1+\nu}\sum_{j=0}^k\left(\frac{2}{3}\right)^j
\end{aligned}
$$

for some positive constant M depending on ν. By checking carefully the proof of Lemma 3.4, we see that the estimates in (3.100) are still valid.

3.2.4. The existence and uniqueness of solutions. In view of Lemma 3.4, it is obvious that the sequences $(W^{(k)}, R^{(k)}, S^{(k)})(\xi,\eta)$ converge uniformly and the limit functions, denoted by (W, R, S), are continuous. Moreover, by Lemma 3.2, the functions (W, R, S) satisfy

$$|W(\xi,\eta)|; |R(\xi,\eta)|; |S(\xi,\eta)| \le 3M\xi^2, \quad |R(\xi,\eta) - S(\xi,\eta)| \le 3M\xi^2 \tag{3.129}$$

for any $(\xi,\eta) \in \overline{D}_\delta$. It is easily seen that the functions (W, R, S) also satisfy the system of integral equations (3.45) and the initial conditions $W(0,\eta) = R(0,\eta) = S(0,\eta) = 0$. In addition, from the relation between ξ_i and ξ, η

$$\eta - \int_{\xi_i}^{\xi} \lambda_i(t, z_i(t;\xi,\eta))\ \mathrm{d}t = \tilde{z}(\xi_i) = \int_0^{\xi_i} \lambda_-(t, \tilde{z}(t))\ \mathrm{d}t, \quad i = 0, +, \tag{3.130}$$

it suggests that $\eta = \tilde{z}(\xi_i)$ iff $\xi = \xi_i$ for $i = 0, +$. Thus by (3.45) and (3.129) we have $W(\xi_0, \tilde{z}(\xi_0)) = h(\xi_0)$ and $R(\xi_+, \tilde{z}(\xi_+)) = b(\xi_+)$, which mean that the functions (W, R) satisfy the boundary conditions in (3.23). The boundary condition $S(t, \tilde{z}(t)) = s(t)$ in (3.23) follows directly from (3.31).

We next check the initial conditions $W_\xi(0,\eta) = R_\xi(0,\eta) = S_\xi(0,\eta) = 0$. To this end, we shall show that the limits (W, R, S) obtained with the aid of Lemma 3.4 are C^1 functions. According to the forms of integral equations (3.45), it is easy to see that $S(\xi,\eta)$ possesses one continuous derivative with respect to ξ and then $S_\xi(0,\eta) = 0$. For the functions W and R, we need derive estimates of $\xi_{i\xi}$ $(i = 0, +)$. Differentiating (3.130) with respect to ξ leads to

$$\xi_{i\xi} = \frac{\lambda_i(\xi,\eta) + \displaystyle\int_{\xi_i}^{\xi} \frac{\partial \lambda_i}{\partial z} \cdot \frac{\partial z_i}{\partial \xi}\ \mathrm{d}t}{\lambda_i(\xi_i, \tilde{z}(\xi_i)) - \lambda_-(\xi_i, \tilde{z}(\xi_i))}, \quad \frac{\partial z_i}{\partial \xi}(t;\xi,\eta) = -\lambda_i \frac{\partial z_i}{\partial \eta}(t;\xi,\eta),$$

which along with (3.40) and (3.129) lead to

$$\xi_{0\xi} \le \frac{K_0}{\xi}, \quad \xi_{+\xi} \le \frac{K_0}{\xi^2},$$

from which and (3.115), (3.121) we know that $W(\xi,\eta)$ and $S(\xi,\eta)$ possess one continuous derivative with respect to ξ and then $W_\xi(0,\eta) = R_\xi(0,\eta) = 0$.

To establish the existence of (W_η, R_η, S_η) in $\overline{D}_\delta$, we consider the following linear system of integral equations obtained by differentiating (3.45) with respect to η:

$$\begin{cases} W_\eta(\xi,\eta) = g_1(\xi_0) + \displaystyle\int_{\xi_0}^{\xi} \Big\{ C_{11}tR_z + C_{12}tS_z + C_{13}t \Big\} \frac{\partial z_0}{\partial \eta}(t, z_0(t))\ \mathrm{d}t, \\ R_\eta(\xi,\eta) = g_2(\xi_+) + \displaystyle\int_{\xi_+}^{\xi} \Big\{ \frac{R_z - S_z}{2t} + 2A_1 \frac{(R-S)(R_z - S_z)}{t} \\ \qquad + A_{11}R_z + A_{12}S_z + A_{13}W_z + A_{14} + Tt \Big\} \frac{\partial z_+}{\partial \eta}(t, z_+(t))\ \mathrm{d}t, \\ S_\eta(\xi,\eta) = \displaystyle\int_0^{\xi} \Big\{ \frac{S_z - R_z}{2t} + 2B_1 \frac{(R-S)(R_z - S_z)}{t} + B_{11}S_z \\ \qquad + B_{12}R_z + B_{13}W_z + B_{14} + Tt \Big\} \frac{\partial z_-}{\partial \eta}(t, z_-(t))\ \mathrm{d}t. \end{cases} \tag{3.131}$$

The coefficient functions in (3.131) are given in (3.68) but with the limit functions (W,R,S,z_i) replacing $(W^{(k)},R^{(k)},S^{(k)},z_i^{(k)})$. Clearly, by (3.129), the estimates (3.79)–(3.82) are conservative. Let $g_3(t)$ be the solution of the following ODE problem:

$$\begin{cases} \dfrac{\mathrm{d}g_3}{\mathrm{d}t} = \dfrac{g_3-g_2}{2t} + 2B_1\dfrac{(R-S)(g_2-g_3)}{t} + B_{11}g_3 + B_{12}g_2 + B_{13}g_1 + B_{14} + Tt, \\ g_3(0) = g_3'(0) = 0. \end{cases}$$

The solvability of the above ODE problem can be derived as in Lemma 3.1. Furthermore, the function $g_3(t)$ satisfies

$$g_3(t) \le K_0 t^2, \quad g_3'(t) \le K_0 t. \tag{3.132}$$

Now, for the integral system (3.131), we set $(\widetilde{W}_z^{(0)},\widetilde{R}_z^{(0)},\widetilde{S}_z^{(0)}) = (g_1(t),g_2(t),g_3(t))$ and construct the sequence of vector functions $(\widetilde{W}_z^{(k)},\widetilde{R}_z^{(k)},\widetilde{S}_z^{(k)})(k\ge 1)$ as follows:

$$\begin{cases} \widetilde{W}_\eta^{(k+1)}(\xi,\eta) = g_1(\xi_0) + \displaystyle\int_{\xi_0}^{\xi}\Big\{C_{11}t\widetilde{R}_z^{(k)} + C_{12}t\widetilde{S}_z^{(k)} + C_{13}t\Big\}\frac{\partial z_0^{(k)}}{\partial\eta}(t,z_0(t))\,\mathrm{d}t, \\ \widetilde{R}_\eta^{(k+1)}(\xi,\eta) = g_2(\xi_+) + \displaystyle\int_{\xi_+}^{\xi}\Big\{\frac{\widetilde{R}_z^{(k)}-\widetilde{S}_z^{(k)}}{2t} + 2A_1\frac{(R-S)(\widetilde{R}_z^{(k)}-\widetilde{S}_z^{(k)})}{t} \\ \qquad\qquad +A_{11}\widetilde{R}_z^{(k)} + A_{12}\widetilde{S}_z^{(k)} + A_{13}\widetilde{W}_z^{(k)} + A_{14} + Tt\Big\}\dfrac{\partial z_+^{(k)}}{\partial\eta}(t,z_+(t))\,\mathrm{d}t, \\ \widetilde{S}_\eta^{(k+1)}(\xi,\eta) = \displaystyle\int_0^{\xi}\Big\{\frac{\widetilde{S}_z^{(k)}-\widetilde{R}_z^{(k)}}{2t} + 2B_1\frac{(R-S)(\widetilde{R}_z^{(k)}-\widetilde{S}_z^{(k)})}{t} + B_{11}\widetilde{S}_z^{(k)} \\ \qquad\qquad +B_{12}\widetilde{R}_z^{(k)} + B_{13}\widetilde{W}_z^{(k)} + B_{14} + Tt\Big\}\dfrac{\partial z_-^{(k)}}{\partial\eta}(t,z_-(t))\,\mathrm{d}t, \end{cases} \tag{3.133}$$

where

$$\frac{\partial z_0^{(k)}}{\partial\eta} = \exp\Big\{\int_\xi^t\Big(C_{14}\widetilde{R}_z^{(k)} + C_{15}\widetilde{S}_z^{(k)} + C_{16}\Big)(t,z_0(t))\,\mathrm{d}t\Big\},$$
$$\frac{\partial z_+^{(k)}}{\partial\eta} = \exp\Big\{\int_\xi^t\Big(A_{15}\widetilde{S}_z^{(k)} + A_{16}\widetilde{W}_z^{(k)} + A_{17}\Big)(t,z_+(t))\,\mathrm{d}t\Big\},$$
$$\frac{\partial z_-^{(k)}}{\partial\eta} = \exp\Big\{\int_\xi^t\Big(B_{15}\widetilde{R}_z^{(k)} + B_{16}\widetilde{W}_z^{(k)} + B_{17}\Big)(t,z_-(t))\,\mathrm{d}t\Big\}.$$

For the sequence $(\widetilde{W}_\eta^{(k)},\widetilde{R}_\eta^{(k)},\widetilde{S}_\eta^{(k)})(k\ge 0)$, we first have the following.

LEMMA 3.5. *For all $k\ge 1$ the inequalities*

$$\begin{aligned} &\big|\widetilde{W}_\eta^{(k)}(\xi,\eta)\big| \le M\xi\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j, \quad \big|\widetilde{R}_\eta^{(k)}(\xi,\eta)\big|;\big|\widetilde{S}_\eta^{(k)}(\xi,\eta)\big| \le M\xi^2\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j, \\ &\big|\widetilde{R}_\eta^{(k)}(\xi,\eta) - \widetilde{S}_\eta^{(k)}(\xi,\eta)\big| \le M\xi^2\sum_{j=0}^{k}\Big(\frac{2}{3}\Big)^j \end{aligned} \tag{3.134}$$

hold in $\overline{D}_\delta$.

Proof. The proof is completely similar to the proof of Lemma 3.3 and hence is omitted here. ∎

We next establish a lemma to derive the uniform convergence of the sequence $(\widetilde{W}_\eta^{(k)}, \widetilde{R}_\eta^{(k)}, \widetilde{S}_\eta^{(k)})(k \geq 0)$.

LEMMA 3.6. *For all $k \geq 0$ the inequalities*

$$\begin{aligned}
&\left|\widetilde{W}_\eta^{(k+1)}(\xi,\eta) - \widetilde{W}_\eta^{(k)}(\xi,\eta)\right| \leq M\xi\left(\frac{2}{3}\right)^k, \\
&\left|\widetilde{R}_\eta^{(k+1)}(\xi,\eta) - \widetilde{R}_\eta^{(k)}(\xi,\eta)\right|; \left|\widetilde{S}_\eta^{(k+1)}(\xi,\eta) - \widetilde{S}_\eta^{(k)}(\xi,\eta)\right| \leq M\xi^2\left(\frac{2}{3}\right)^k
\end{aligned} \tag{3.135}$$

hold in $\overline{D}_\delta$.

Proof. We proceed by induction again. For $n = 0$, we have by (3.26), (3.79), (3.80), (3.82), and (3.132)

$$\begin{aligned}
&\left|\widetilde{W}_\eta^{(1)}(\xi,\eta) - \widetilde{W}_\eta^{(0)}(\xi,\eta)\right| \\
&\leq |g_1(\xi_0)| + |g_1(\xi)| + \int_0^\xi \left\{t|C_{11}|\cdot|g_2| + t|C_{12}|\cdot|g_3| + |C_{13}|t\right\}\left|\frac{\partial z_0^{(0)}}{\partial\eta}\right| \,\mathrm{d}t \\
&\leq 2K_0\xi + \int_0^\xi (2M^2t^2 + Mt)\exp(2M\delta^2)\,\mathrm{d}t \\
&\leq M\xi\left(\frac{1}{2} + (M^2\delta^2 + \delta)\exp(2M\delta^2)\right) \leq M\xi
\end{aligned}$$

and

$$\begin{aligned}
\left|\widetilde{R}_\eta^{(1)}(\xi,\eta) - \widetilde{R}_\eta^{(0)}(\xi,\eta)\right| &\leq |g_2(\xi_0)| + |g_2(\xi)| + \int_0^\xi \left\{\frac{|g_2 - g_3|}{2t} + 2M\cdot 3Mt|g_2 - g_3|\right. \\
&\quad \left. + M|g_2| + 2M|g_3| + 2Mt|g_1| + 12M^2t^2 + K_0t\right\}\left|\frac{\partial z_+^{(0)}}{\partial\eta}\right| \,\mathrm{d}t \\
&\leq 2K_0\xi^2 + \int_0^\xi (2K_0t + 3M^3t^3 + 14M^2t^2)\exp(2M\delta^2)\,\mathrm{d}t \\
&\leq M\xi^2\left\{\frac{1}{2} + \left(\frac{1}{4} + 6M\delta\right)\exp(2M\delta^2)\right\} \leq M\xi^2,
\end{aligned}$$

which indicate that each of the inequalities in (3.135) holds for $n = 0$.

Assume (3.135) holds for $n = k-1$. Then for $n = k$, one obtains

$$\left|\widetilde{W}_\eta^{(k+1)}(\xi,\eta) - \widetilde{W}_\eta^{(k)}(\xi,\eta)\right| \leq I_{28} + I_{29}, \tag{3.136}$$

where

$$\begin{aligned}
I_{28} &= \int_{\xi_0}^\xi \left\{t|C_{11}|\cdot\left|\widetilde{R}_z^{(k)} - \widetilde{R}_z^{(k-1)}\right| + t|C_{12}|\cdot\left|\widetilde{S}_z^{(k)} - \widetilde{S}_z^{(k-1)}\right|\right\}\left|\frac{\partial z_0^{(k)}}{\partial\eta}\right| \,\mathrm{d}t, \\
I_{29} &= \int_{\xi_0}^\xi \left\{t|C_{11}|\cdot\left|\widetilde{R}_z^{(k-1)}\right| + t|C_{12}|\cdot\left|\widetilde{S}_z^{(k-1)}\right| + t|C_{13}|\right\}\left|\frac{\partial z_0^{(k)}}{\partial\eta} - \frac{\partial z_0^{(k-1)}}{\partial\eta}\right| \,\mathrm{d}t.
\end{aligned}$$

It is easily checked by (3.134) and the induction assumptions that

$$I_{28} \leq \int_0^\xi 2M \cdot Mt^2 \left(\frac{2}{3}\right)^{k-1} \exp(2M\delta^2)\ \mathrm{d}t \leq M\xi \cdot M\delta^2 \exp(2M\delta^2)\left(\frac{2}{3}\right)^{k-1}$$

and

$$\begin{aligned} I_{29} \leq & \int_0^\xi (6M^2t^2 + Mt)\exp(2M\delta^2) \\ & \quad \times \int_0^\delta \left(M\left|\widetilde{R}_z^{(k)} - \widetilde{R}_z^{(k-1)}\right| + M\left|\widetilde{S}_z^{(k)} - \widetilde{S}_z^{(k-1)}\right|\right) \mathrm{d}s\ \mathrm{d}t \\ \leq & \int_0^\xi (6M^2t^2 + Mt)\exp(2M\delta^2)\cdot M^2\delta^3\left(\frac{2}{3}\right)^{k-1} \mathrm{d}t \\ \leq & M\xi \cdot M\delta^2 \exp(2M\delta^2)\left(\frac{2}{3}\right)^{k-1}. \end{aligned}$$

Putting the above into (3.136) gives

$$\left|\widetilde{W}_\eta^{(k+1)}(\xi,\eta) - \widetilde{W}_\eta^{(k)}(\xi,\eta)\right| \leq M\xi \cdot 2M\delta^2 \exp(2M\delta^2)\left(\frac{2}{3}\right)^{k-1} \leq M\xi\left(\frac{2}{3}\right)^{k}. \tag{3.137}$$

For the function $\widetilde{R}_z$, we have similarly

$$\left|\widetilde{R}_\eta^{(k+1)}(\xi,\eta) - \widetilde{R}_\eta^{(k)}(\xi,\eta)\right| \leq I_{30} + I_{31}, \tag{3.138}$$

where

$$\begin{aligned} I_{30} = & \int_{\xi_+}^\xi \left\{ \frac{\left|\widetilde{R}_z^{(k)} - \widetilde{R}_z^{(k-1)}\right| + \left|\widetilde{S}_z^{(k)} - \widetilde{S}_z^{(k-1)}\right|}{2t} \right. \\ & \quad + 6M^2t(\left|\widetilde{R}_z^{(k)} - \widetilde{R}_z^{(k-1)}\right| + \left|\widetilde{S}_z^{(k)} - \widetilde{S}_z^{(k-1)}\right|) + M\left|\widetilde{R}_z^{(k)} - \widetilde{R}_z^{(k-1)}\right| \\ & \quad \left. + 2M\left|\widetilde{S}_z^{(k)} - \widetilde{S}_z^{(k-1)}\right| + 2Mt\left|\widetilde{W}_z^{(k)} - \widetilde{W}_z^{(k-1)}\right| \right\} \left|\frac{\partial z_+^{(k)}}{\partial \eta}\right| \mathrm{d}t, \\ I_{31} = & \int_{\xi_+}^\xi \left\{ \frac{\left|\widetilde{R}_z^{(k-1)} - \widetilde{S}_z^{(k-1)}\right|}{2t} + 2|A_1|\frac{|R-S|\cdot\left|\widetilde{R}_z^{(k-1)} - \widetilde{S}_z^{(k-1)}\right|}{t} + |A_{11}|\cdot\left|\widetilde{R}_z^{(k-1)}\right| \right. \\ & \left. + |A_{12}|\cdot\left|\widetilde{S}_z^{(k-1)}\right| + |A_{13}|\cdot\left|\widetilde{W}_z^{(k-1)}\right| + |A_{14}| + |T|t \right\} \left|\frac{\partial z_+^{(k)}}{\partial \eta} - \frac{\partial z_+^{(k-1)}}{\partial \eta}\right| \mathrm{d}t. \end{aligned}$$

Using the same arguments as led to I_{28} and I_{29} yield

$$\begin{aligned} I_{30} \leq & \int_0^\xi \left\{ Mt\left(\frac{2}{3}\right)^{k-1} + 12M^3t^3\left(\frac{2}{3}\right)^{k-1} + 5M^2t^2\left(\frac{2}{3}\right)^{k-1} \right\} \exp(2M\delta^2)\ \mathrm{d}t \\ \leq & M\xi^2\left(\frac{1}{2} + 3M^2\delta^2 + 2M\delta\right)\exp(2M\delta^2)\left(\frac{2}{3}\right)^{k-1} \end{aligned}$$

and

$$
\begin{aligned}
I_{31} \le & \int_0^\xi \left\{\frac{3}{2}Mt + 18M^3t^3 + 27M^2t^2 + Mt\right\} \\
& \qquad \times \exp(2M\delta^2)\int_0^\delta Ms^2\left(|\widetilde{S}_z^{(k)} - \widetilde{S}_z^{(k-1)}| + |\widetilde{W}_z^{(k)} - \widetilde{W}_z^{(k-1)}|\right)\,\mathrm{d}s\,\mathrm{d}t \\
\le & \int_0^\xi \left\{\frac{5}{2}Mt + 28M^2t^2\right\}\exp(2M\delta^2)\cdot M^2\delta^4\left(\frac{2}{3}\right)^{k-1}\mathrm{d}t \\
\le & \; M\xi^2\delta\exp(2M\delta)\left(\frac{2}{3}\right)^{k-1}.
\end{aligned}
$$

From the above, we finally acquire

$$
\begin{aligned}
\left|\widetilde{R}_\eta^{(k+1)}(\xi,\eta) - \widetilde{R}_\eta^{(k)}(\xi,\eta)\right| &\le M\xi^2\left(\frac{1}{2} + 3M\delta + \delta\right)\exp(2M\delta^2)\left(\frac{2}{3}\right)^{k-1} \\
&\le M\xi^2\left(\frac{2}{3}\right)^k
\end{aligned} \tag{3.139}
$$

by the choice of δ in (3.50). This estimate is also valid for the function $\widetilde{S}_\eta$. We combine (3.137) and (3.139) to end the proof of the lemma. ∎

Based on Lemmas 3.5 and 3.6, we know that the sequences $(\widetilde{W}_\eta^{(k)}, \widetilde{R}_\eta^{(k)}, \widetilde{S}_\eta^{(k)})(\xi,\eta)$ converge uniformly, which indicates that the functions $(W_\eta, R_\eta, S_\eta)(\xi,\eta)$ are continuous and satisfy

$$
\begin{aligned}
&\left|W_\eta(\xi,\eta)\right| \le 3M\xi, \quad \left|R_\eta(\xi,\eta)\right|; \left|S_\eta(\xi,\eta)\right| \le 3M\xi^2, \\
&\left|R_\eta(\xi,\eta) - S_\eta(\xi,\eta)\right| \le 3M\xi^2.
\end{aligned} \tag{3.140}
$$

Since the functions (W, R, S) satisfy (3.45) and have the required differentiability properties, it is the smooth solution of (3.27) satisfying mixed-type boundary conditions (3.23).

For the uniqueness, we consider the difference of solutions. Let (W_1, R_1, S_1) and (W_2, R_2, S_2) be two smooth solutions of (3.27). Denote $\widehat{W} = W_2 - W_1$, $\widehat{R} = R_2 - R_1$ and $\widehat{S} = S_2 - S_1$. Then, by (3.49), (3.81), (3.129), and (3.140), $(\widehat{W}, \widehat{R}, \widehat{S})$ satisfy the homogeneous integral inequality system

$$
\begin{cases}
\left|\widehat{W}(\xi,\eta)\right| \le \widehat{M}\displaystyle\int_0^\xi \left(|\widehat{R}| + |\widehat{S}|\right)\mathrm{d}t, \\
\left|\widehat{R}(\xi,\eta)\right| \le \displaystyle\int_0^\xi \left\{\frac{|\widehat{R}-\widehat{S}|}{2t} + \widehat{M}\left(|\widehat{W}| + \widehat{R}| + |\widehat{S}|\right)\right\}\mathrm{d}t, \\
\left|\widehat{S}(\xi,\eta)\right| \le \displaystyle\int_0^\xi \left\{\frac{|\widehat{R}-\widehat{S}|}{2t} + \widehat{M}\left(|\widehat{W}| + \widehat{R}| + |\widehat{S}|\right)\right\}\mathrm{d}t, \\
\left|\widehat{R}(\xi,\eta) - \widehat{S}(\xi,\eta)\right| \le \displaystyle\int_0^\xi \left\{\frac{|\widehat{R}-\widehat{S}|}{t} + \widehat{M}\left(|\widehat{W}| + \widehat{R}| + |\widehat{S}|\right)\right\}\mathrm{d}t
\end{cases} \tag{3.141}
$$

for some positive constant $\widehat{M}$. By repeating the insertion of these in the right side of (3.141), one can acquire that the functions $(\widehat{W}, \widehat{R}, \widehat{S})$ must satisfy the inequalities of the forms

$$
\left|\widehat{W}\right|; \left|\widehat{R}\right|; \left|\widehat{S}\right| \le M^*\left(\frac{2}{3}\right)^k
$$

for arbitrary k and some positive constant M^*, which says that there holds $\widehat{W} = \widehat{R} = \widehat{S} \equiv 0$.

Finally, we note by (3.22) that the problem (3.27), (3.23) is equivalent to the problem (3.5), (3.14). Hence the proof of Theorem 3.1 is complete.

4. Solutions in terms of physical variables. In view of the previous section, the mixed-type boundary value problem (3.5), (3.14) admits a unique local classical solution $(H, U, V)(t, r)$ in the region $\overline{D}' := \{(t, r)|\ t \in [0, \delta], \tilde{r}(\delta_2) - K\delta_2^3 + Kt^3 \le r \le \tilde{r}(t)\}$. In this section, we convert this solution in the partial hodograph plane to that in the original physical plane to construct a classical solution for Problem 1.

We first recall the coordinate transformation (3.1) to obtain

$$\frac{\partial x}{\partial t} = \frac{\theta_y}{J}, \quad \frac{\partial y}{\partial t} = -\frac{\theta_x}{J}, \quad \frac{\partial x}{\partial r} = \frac{\tan\omega\varpi_y}{J}, \quad \frac{\partial y}{\partial r} = -\frac{\tan\omega\varpi_x}{J}, \tag{4.1}$$

where J is the Jacobian defined in (3.2), and

$$\begin{aligned} \theta_x &= (t\sin r - \sqrt{1-t^2}\cos r)U + (t\sin r + \sqrt{1-t^2}\cos r)V, \\ \theta_y &= -(t\cos r + \sqrt{1-t^2}\sin r)U - (t\cos r - \sqrt{1-t^2}\sin r)V, \\ \varpi_x &= -(\kappa+1-t^2)[\sin r(U-V) - \sqrt{1-t^2}\cos r\tfrac{U+V}{t} + 2\sin rG_1H], \\ \varpi_y &= (\kappa+1-t^2)[\cos r(U-V) + \sqrt{1-t^2}\sin r\tfrac{U+V}{t} + 2\cos rG_1H]. \end{aligned} \tag{4.2}$$

From (4.1) and (4.2), one finds that

$$\begin{aligned} x_t &= -\frac{(t\cos r + \sqrt{1-t^2}\sin r)U(t,r) + (t\cos r - \sqrt{1-t^2}\sin r)V(t,r)}{2F(t)\{2U(t,r)V(t,r) + G_1(t)H(t,r)[V(t,r) - U(t,r)]\}}t, \\ y_t &= -\frac{(t\sin r - \sqrt{1-t^2}\cos r)U(t,r) + (t\sin r + \sqrt{1-t^2}\cos r)V(t,r)}{2F(t)\{2U(t,r)V(t,r) + G_1(t)H(t,r)[V(t,r) - U(t,r)]\}}t \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} x_r &= \frac{\cos r(U(t,r) - V(t,r)) + \sqrt{1-t^2}\sin r\frac{U(t,r)+V(t,r)}{t} + 2\cos rG_1H(t,r)}{2\sqrt{1-t^2}\{2U(t,r)V(t,r) + G_1H(t,r)[V(t,r) - U(t,r)]\}}, \\ y_r &= \frac{\sin r(U(t,r) - V(t,r)) - \sqrt{1-t^2}\cos r\frac{U(t,r)+V(t,r)}{t} + 2\sin rG_1H(t,r)}{2\sqrt{1-t^2}\{2U(t,r)V(t,r) + G_1H(t,r)[V(t,r) - U(t,r)]\}}. \end{aligned} \tag{4.4}$$

Let $r = r_-(t)$ be the negative characteristics of (3.5) defined by

$$\frac{\mathrm{d}r_-(t)}{\mathrm{d}t} = -\frac{\sqrt{1-t^2}V(t, r_-(t))t^2}{F(t)[V(t, r_-(t)) - G_1(t)H(t, r_-(t))]}. \tag{4.5}$$

Then using (4.3)–(4.4) and doing simplifications, we have

$$\begin{aligned} \frac{\mathrm{d}x(t, r_-(t))}{\mathrm{d}t} &= -\frac{t\cos r + \sqrt{1-t^2}\sin r}{2F(t)[V(t,r) - G_1H(t,r)]}t, \\ \frac{\mathrm{d}y(t, r_-(t))}{\mathrm{d}t} &= -\frac{t\sin r - \sqrt{1-t^2}\cos r}{2F(t)[V(t,r) - G_1H(t,r)]}t. \end{aligned} \tag{4.6}$$

Moreover, by (4.5), the region $\overline{D}'$ is divided into two subregions by the negative characteristic $r=\bar{r}(t)$ defined by

$$\begin{cases} \dfrac{\mathrm{d}\bar{r}(t)}{\mathrm{d}t} = -\dfrac{\sqrt{1-t^2}V(t,r)t^2}{F(t)[V(t,r)-G_1(t)H(t,r)]}, \\ \bar{r}(0)=r_2. \end{cases}$$

Denote $\overline{D}'_- = \overline{D}' \cap \{r \le \bar{r}(t)\}$ and $\overline{D}'_+ = \overline{D}' \cap \{r > \bar{r}(t)\}$. For any point $(\hat{t},\hat{r}) \in \overline{D}'$, we know that the negative characteristic $r_-(t;\hat{t},\hat{r})$ intersects the line $t=0$ if $(\hat{t},\hat{r}) \in \overline{D}'_-$ and intersects the curve $\widehat{B'A'}$ if $(\hat{t},\hat{r}) \in \overline{D}'_+$. We now employ (4.6) to define the value $\hat{x} = x(\hat{t},\hat{r})$,

(4.7)

$$x(\hat{t},\hat{r}) = \begin{cases} \hat{\theta}^{-1}(\check{r}) \\ \quad -\displaystyle\int_0^{\hat{t}} \frac{t\cos r_-(t;\hat{t},\hat{r}) + \sqrt{1-t^2}\sin r_-(t;\hat{t},\hat{r})}{2F(t)[V(t,r_-(t;\hat{t},\hat{r})) - G_1(t)H(t,r_-(t;\hat{t},\hat{r}))]} t\,\mathrm{d}t, \ (\hat{t},\hat{r}) \in \overline{D}'_-, \\ \psi^{-1}(\tilde{\theta}^{-1}(\tilde{r})) \\ \quad -\displaystyle\int_{\tilde{t}}^{\hat{t}} \frac{t\cos r_-(t;\hat{t},\hat{r}) + \sqrt{1-t^2}\sin r_-(t;\hat{t},\hat{r})}{2F(t)[V(t,r_-(t;\hat{t},\hat{r})) - G_1(t)H(t,r_-(t;\hat{t},\hat{r}))]} t\,\mathrm{d}t, \ (\hat{t},\hat{r}) \in \overline{D}'_+, \end{cases}$$

where $\hat{\theta}^{-1}$, $\tilde{\theta}^{-1}$, and ψ^{-1} represent, respectively, the inverses of $\hat{\theta}$, $\tilde{\theta}$, and ψ, and the constant $\check{r}$ is given by

$$\check{r} = \hat{r} - \int_0^{\hat{t}} \frac{\sqrt{1-t^2}V(t,r_-(t,\hat{t},\hat{r}))t^2}{F(t)[V(t,r_-(t,\hat{t},\hat{r})) - G_1(t)H(t,r_-(t,\hat{t},\hat{r}))]}\,\mathrm{d}t.$$

The numbers $\tilde{t}$ and $\tilde{r}$ in (4.7) are determined by the following equations:

$$\begin{cases} \tilde{r} = r_2 + \displaystyle\int_0^{\tilde{r}} \frac{\sqrt{1-t^2}\tilde{b}_0(t)t^2}{F(t)[\tilde{b}_0(t) + G_1(t)\tilde{h}_0(t)]}\,\mathrm{d}t, \\ \tilde{r} = \hat{r} + \displaystyle\int_{\tilde{t}}^{\hat{r}} \frac{\sqrt{1-t^2}V(t,r_-(t,\hat{t},\hat{r}))t^2}{F(t)[V(t,r_-(t,\hat{t},\hat{r})) - G_1(t)H(t,r_-(t,\hat{t},\hat{r}))]}\,\mathrm{d}t. \end{cases}$$

We point out that the point $(\tilde{t},\tilde{r})$ is the intersection of the positive characteristic $r=\tilde{r}(t)$ and the negative characteristic $r=r_-(t;\hat{t},\hat{r})$. With the same argument, we can also define the value $\hat{y} = y(\hat{t},\hat{r})$:

(4.8)

$$y(\hat{t},\hat{r}) = \begin{cases} \varphi(\hat{\theta}^{-1}(\check{r})) \\ \quad -\displaystyle\int_0^{\hat{t}} \frac{t\sin r_-(t;\hat{t},\hat{r}) - \sqrt{1-t^2}\cos r_-(t;\hat{t},\hat{r})}{2F(t)[V(t,r_-(t;\hat{t},\hat{r})) - G_1(t)H(t,r_-(t;\hat{t},\hat{r}))]} t\,\mathrm{d}t, \ (\hat{t},\hat{r}) \in \overline{D}'_-, \\ \tilde{\theta}^{-1}(\tilde{r}) \\ \quad -\displaystyle\int_{\tilde{t}}^{\hat{t}} \frac{t\sin r_-(t;\hat{t},\hat{r}) - \sqrt{1-t^2}\cos r_-(t;\hat{t},\hat{r})}{2F(t)[V(t,r_-(t;\hat{t},\hat{r})) - G_1(t)H(t,r_-(t;\hat{t},\hat{r}))]} t\,\mathrm{d}t, \qquad (\hat{t},\hat{r}) \in \overline{D}'_+. \end{cases}$$

Therefore, corresponding to the region $\overline{D}'$ in the (t,r)-plane, we obtain the region $\overline{D}$ in the original (x,y)-plane

$$\overline{D} = \{(x,y) |\ x = x(t,r), y = y(t,r), (t,r) \in \overline{D}'\}.$$

In addition, one can use (4.1) to calculate the Jacobian of the map $(t,r)\mapsto(x,y)$

$$j:=\frac{\partial(x,y)}{\partial(t,r)}=\frac{t}{2F(t)\{2U(t,r)V(t,r)+G_1H(t,r)[V(t,r)-U(t,r)]\}},$$

which is strictly less than zero in $\overline{D}'\setminus\{t=0\}$. Thus the map $(t,r)\mapsto(x,y)$ is a local one-to-one mapping for $t\in(0,\delta]$. Moreover, we claim that this mapping is global one-to-one, including the line $t=0$. To confirm the claim, we note by (3.129) and the fact $(U+V)/t=(R-S)/t+2\hat{a}_1$ that the functions x_t,y_t,x_r,y_r are well-defined up to the line $t=0$. In addition, it follows by (3.129), (2.30), and the assumptions in (2.32) that

$$U(t,r)>0,\quad V(t,r)<0,$$
$$\cos r(U-V)+\sqrt{1-t^2}\sin r\frac{U+V}{t}+2\cos rG_1H>0,$$

on $\overline{D}'$, which imply by (4.3) and (4.4) that $x_r<0$, $y_r<0$ and $x_t\geq 0$, $y_t\leq 0$ with $y_t=0$ and $x_t=0$ iff $t=0$. Thanks to the above properties, for any two different points (t_1,r_1) and (t_2,r_2) in $\overline{D}'$, if $t_1=t_2,r_1<r_2$ or $r_1=r_2,t_1<t_2$, then $y(t_1,r_1)>y(t_2,r_2)$; if $t_1<t_2,r_1<r_2$, then $y(t_1,r_1)<y(t_2,r_2)$; if $t_1<t_2,r_1>r_2$, then $x(t_1,r_1)<x(t_2,r_2)$. This means that for any different points (t_1,r_1) and (t_2,r_2) in $\overline{D}'$, the images $(x(t_1,r_1),y(t_1,r_1))$ and $(x(t_2,r_2),y(t_2,r_2))$ are different. Thus we have established the global one-to-one property of the mapping $(t,r)\mapsto(x,y)$. Therefore, for any point $(x^*,y^*)\in\overline{D}$, there exists a unique corresponding point $(t^*,r^*)\in\overline{D}'$. Now, we can construct the functions $(\theta,\varpi,H)(x,y)$ as follows:

$$\theta(x^*,y^*)=r^*,\ \varpi(x^*,y^*)=\sqrt{1-(t^*)^2},$$
$$H(x^*,y^*)=H(t^*,r^*)\quad\forall\ (x^*,y^*)\in\overline{D}. \tag{4.9}$$

We next check that the functions $(\theta,\varpi,H)(x,y)$ defined in (4.9) are a solution of system (2.12). By the construction of $x(t,r)$ and $y(t,r)$ in (4.7)–(4.8), we can obtain that the functions $(\theta,\varpi,H)(x,y)$ satisfy the boundary conditions. Indeed, for any point $(x,\varphi(x))$ on $\widehat{BC}$, it is easy to see that $(\hat{t},\hat{r})=(0,\hat{\theta}(x))$ satisfies the following equation:

$$\varphi\bigg(\hat{\theta}^{-1}(\check{r})-\int_0^{\hat{t}}\frac{t\cos r_-(t;\hat{t},\hat{r})+\sqrt{1-t^2}\sin r_-(t;\hat{t},\hat{r})}{2F(t)[V(t,r_-(t;\hat{t},\hat{r}))-G_1(t)H(t,r_-(t;\hat{t},\hat{r}))]}t\ \mathrm{d}t\bigg)$$
$$=\varphi(\hat{\theta}^{-1}(\check{r}))-\int_0^{\hat{t}}\frac{t\sin r_-(t;\hat{t},\hat{r})-\sqrt{1-t^2}\cos r_-(t;\hat{t},\hat{r})}{2F(t)[V(t,r_-(t;\hat{t},\hat{r}))-G_1(t)H(t,r_-(t;\hat{t},\hat{r}))]}t\ \mathrm{d}t. \tag{4.10}$$

Since the mapping $(t,r)\mapsto(x,y)$ is global one-to-one, then $(0,\hat{\theta}(x))$ is the unique solution of (4.10). Hence we have $\varpi(x,\varphi(x))=1$ and $\theta(x,\varphi(x))=\hat{\theta}(x)$. Other boundary conditions can be checked similarly. Moreover, the functions $(\theta,\varpi)(x,y)$ are uniformly continuous up to the sonic boundary, which follow directly from the uniform boundedness of $\partial_x r,\partial_y r,\partial_x\sqrt{1-t^2},\partial_y\sqrt{1-t^2}$ by (4.2). For the function $H(x,y)$, we can derive the expressions of H_x,H_y,

$$H_x=-2\sin r\bigg(\frac{(U-V)(U+G_1H)}{\frac{R-S}{t}+2\hat{a}_1}-Ut\bigg)H_r,$$
$$H_y=2\cos r\bigg(\frac{(U-V)(U+G_1H)}{\frac{R-S}{t}+2\hat{a}_1}-Ut\bigg)H_r,$$

which are uniformly bounded by (3.129) and (3.140). Hence the function $H(x,y)$ is also uniformly continuous up to the sonic boundary. We now check that the functions $(\theta,\varpi,H)(x,y)$ satisfy system (2.12). From (4.9) we can define

$$\omega=\arcsin\varpi(x,y),\ \ \alpha=\theta(x,y)+\omega(x,y),\ \ \beta=\theta(x,y)-\omega(x,y)\ \ \forall\ (x,y)\in\overline{D}.$$

Then one calculates by (4.2)

$$\begin{aligned}
&\bar\partial^+\theta+\frac{\cos\omega}{\kappa+\varpi^2}\bar\partial^+\varpi=(\cos\alpha\theta_x+\sin\alpha\theta_y)+\frac{\cos\omega}{\kappa+\varpi^2}(\cos\alpha\varpi_x+\sin\alpha\varpi_y)\\
&=(t\cos r-\sqrt{1-t^2}\sin r)\theta_x+(t\sin r+\sqrt{1-t^2}\cos r)\theta_y\\
&\quad+\frac{t}{\kappa+1-t^2}[(t\cos r-\sqrt{1-t^2}\sin r)\varpi_x+(t\sin r+\sqrt{1-t^2}\cos r)\varpi_y]\\
&=-t(t\cos r-\sqrt{1-t^2}\sin r)\cdot 2\sin rG_1H+t(t\sin r+\sqrt{1-t^2}\cos r)\cdot 2\cos rG_1H\\
&=2t\sqrt{1-t^2}G_1(t)H(t,r)=\sin(2\omega)G(\omega)H(x,y),
\end{aligned}$$

which indicates that the second equation of (2.12) holds. The other two equations in (2.12) can be verified analogously.

We now construct the function $\Omega=\Omega(x,y)$ by solving the linear problem

$$\left\{\begin{array}{l}\cos\alpha(x,y)\Omega_x+\sin\alpha(x,y)\Omega_y=G(\varpi(x,y))H(x,y),\\ \Omega(x,\varphi(x))=\hat\Omega(x),\quad x\in[x_B,x_C).\end{array}\right.\tag{4.11}$$

The solvability of linear problem (4.11) on the region $\overline{D}$ follows directly from the continuity of α,ϖ, and H. Furthermore, the mixed-type boundary conditions (2.18) and (2.19) are satisfied by the construction of the functions $(\theta,\varpi,\Omega)(x,y)$. Hence the proof of Theorem 2.1 is completed. Similarly, we can solve the linear equations

$$\cos\theta(x,y)S_x+\sin\theta(x,y)S_y=0,\quad \cos\theta(x,y)B_x+\sin\theta(x,y)B_y=0$$

subject to the corresponding boundary values to acquire the functions $S(x,y)$ and $B(x,y)$ in $\overline{D}$. Finally, we define the functions $(\rho,u,v,p)(x,y)$ as

$$\begin{aligned}
&c=\sqrt{\frac{2\kappa\varpi^2(x,y)B(x,y)}{\kappa+\varpi^2(x,y)}},\quad u=c(x,y)\frac{\cos\theta(x,y)}{\varpi(x,y)},\quad v=c(x,y)\frac{\sin\theta(x,y)}{\varpi(x,y)},\\
&\rho=\left(\frac{2\kappa B(x,y)\varpi^2(x,y)}{\gamma[\kappa+\varpi^2(x,y)]S(x,y)}\right)^{\frac{1}{\gamma-1}},\quad p=S(x,y)\left(\frac{2\kappa B(x,y)\varpi^2(x,y)}{\gamma[\kappa+\varpi^2(x,y)]S(x,y)}\right)^{\frac{\gamma}{\gamma-1}},
\end{aligned}$$

which is a classical solution to the full Euler equations (1.1) near the corner point B.

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